Norms of Vectors

When we want to measure the length of, or distance between, vectors we need a yardstick that measures in a consistent way, generalizing the idea of absolute value to vector spaces. We can capture the essence of length by requiring that such a yardstick satisfy the following properties:

1. $f(x) > 0, \ \forall x \neq 0$,
2. $f(\alpha x) = |\alpha| f(x), \ \forall x$ and for all scalars α, and
3. $f(x+y) \leq f(x) + f(y), \ \forall x$ and y.

Any function which satisfies these properties can be called a *norm*, and we usually use the symbol $f(x) = \|x\|$. With a norm, $\mathbb{R}^n$ is a metric space, with the distance between x and y being $d(x,y) = \|x - y\|$. The *unit circle* or *unit ball* in $(\mathbb{R}^n, \|\cdot\|)$ is the set of vectors with unit length: $\{x \in \mathbb{R}^n : \|x\| = 1\}$. A norm is completely determined by its unit circle (property 2), so don't be suprised if almost all unit circles aren't... circular.

You have already seen the Euclidean norm, it is also called the 2-norm because it is the $p = 2$ member of this family of p-norms:

$$f(x) \ = \ \|x\|_p \ = \ (\sum_{i=1}^{n} |x_i|^p)^{1/p}.$$

The 1, 2 and ∞ norms are commonly used in computations

$$\|x\|_1 \ = \ \sum_{i=1}^{n} |x_i| \qquad \text{and} \qquad \|x\|_2 \ = \ (\sum_{i=1}^{n} |x_i|^2)^{1/2} \qquad \text{and} \qquad \|x\|_\infty \ = \ \max_{1 \leq i \leq n} |x_i|.$$

The dot product is the standard inner product on $\mathbb{R}^n$, and of course we can write

$$\|x\|_2 = (x^t x)^{1/2}.$$

The *Cauchy-Schwartz inequality* says

$$|x^t y| \leq \|x\|_2 \|y\|_2.$$

There are uncountably many norms on $\mathbb{R}^n$. In fact, if $A \in \mathbb{R}^{m \times n}$ has rank n, and $\|y\|$ is a norm on $\mathbb{R}^m$, then $f(x) = \|Ax\|$ is a norm on $\mathbb{R}^n$.

The natural norm induced by any inner product $<x, y>$ is a 2-norm:

$$\|x\|_2 = (<x, x>)^{1/2},$$

and inner products also give us angles:

$$<x, y> = \|x\|_2 \|y\|_2 \cos(\theta).$$

With the idea of angles we can talk about *orthogonality*:

$$x \perp y \qquad \Longleftrightarrow \qquad < x, y > = 0.$$