The Taylor Polynomial

We like polynomials mostly for their flexibility and simplicity. You know all about the simplicity: they are as smooth as you please, easy to differentiate and integrate, the polynomials of degree less than n form a vector space of dimension n, the product of polynomials is a polynomial, a polynomial of degree n has exactly n roots, etc., etc. An example of the flexibility of polynomials is the

Weierstrass Approximation Theorem:
If f is any function continuous over any finite inverval $[a, b]$, then for any $\epsilon > 0$ there is a polynomial p which satisfies $|p(x) - f(x)| < \epsilon$ for all $x \in [a, b]$.

If you don't yet appreciate this statement, draw a picture for this theorem (with an f that has corners). We even have constructive proofs for this theorem. But this page is about the Taylor polynomial. This polynomial is not so much about approximating on an interval, but rather focusing on a specific point. The statement is as follows:

Taylor Polynomial: If f has $n+1$ continuous derivatives on $[a, b]$, and $x_0 \in [a, b]$, then for each $x \in (a, b)$, there is a $\xi = \xi(x)$ between x and x_0 such that

$$f(x) = P_n(x) + R_n(x), \text{ where}$$

$$P_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j, \quad \text{and} \quad R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{(n+1)}.$$

P_n is the Taylor polynomial for f about x_0, and $R_n(x)$ is its remainder term (or *truncation error* term). Notice that P_n is the polynomial (of degree n or less) that satisfies

$$P_n^{(j)}(x_0) = f^{(j)}(x_0), \quad j = 0, 1, \ldots, n,$$

i.e., at x_0, P_n and its first n derivatives match f and its first n derivatives.

An equivalent way to write P_n is with the parameterization $x = x_0 + h$, giving

$$f(x_0 + h) \quad = \quad f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \cdots + \frac{h^n}{n!}f^{(n)}(x_0) \quad + \quad \frac{h^{n+1}}{(n+1)!}f^{(n+1)}(\xi).$$

Finally, to boil everything down to its essence, if f is smooth enough on $[a, b]$, there is a polynomial P_n, of degree n, such that for all $x \in [a, b]$, (with $h = x - x_0$)

$$f(x) \ = \ P_n(x) \ + \ \mathrm{O}(h^{n+1}).$$