# Interpolating Splines

We have seen that polynomial interpolation may not suit all analyses. One successful generalization is piecewise polynomial approximation. Here, rather than interpolate $n+1$ points by a polynomial of degree $n$, we interpolate the $n+1$ points by a piecewise polynomial function of the form

$$S(x) = \begin{cases} s_0(x) & , \quad x \in [x_0, x_1) \\ s_1(x) & , \quad x \in [x_1, x_2) \\ & \vdots \\ s_{n-1}(x) & , \quad x \in [x_{n-1}, x_n] \end{cases} ,$$

where the $s_j$ are polynomials of (usually) small degree.

A piecewise linear function is a connect-the-dots graph, while a piecewise quadratic function has a parabola between each data point. Now a parabola has 3 degrees of freedom (3 coefficients: it lives in the vector space of polynomials of degree less than 3, which has dimension 3), but there are only 2 interpolatory conditions to be met. So, for each interval there are an infinite number of interpolatory parabolas. What do we do with this freedom? It turns out we can ask that $S$ be smooth, i.e. $S \in C^1([x_0, x_n])$, and still have one degree of freedom left over (you will see how it goes for the cubic case below).

A piecewise cubic function, then, has $4n$ degrees of freedom (4 coefficients for each of the $n$ cubic polynomials), and we can get away with requiring that $S \in C^2([x_0, x_n])$ (continuous curvature). The only place where $S$ might not be smooth is at a node, so our conditions will apply there. Let's see:

1. $s_j(x_j) = y_j, \quad j = 0, 1, \ldots, n-1$ ($n$ interpolation conditions on $S$)
2. $s_j(x_{j+1}) = y_{j+1}, \quad j = 0, 1, \ldots, n-1$ ($n$ $C^0$ continuity conditions on $S$)
3. $s'_j(x_{j+1}) = s'_{j+1}(x_{j+1}), \quad j = 0, 1, \ldots, n-2$ ($n-1$ $C^1$ continuity conditions on $S$)
4. $s''_j(x_{j+1}) = s''_{j+1}(x_{j+1}), \quad j = 0, 1, \ldots, n-2$ ($n-1$ $C^2$ continuity conditions on $S$)

This gives us $4n-2$ conditions. Typically two more boundary (endpoint) conditions are prescribed, giving a uniquely determined *cubic spline interpolator* $S$. Some common boundary conditions are (i) clamped: $S'(x_0) = y'_0$ and $S'(x_n) = y'_n$, (ii) natural: $S''(x_0) = 0 = S''(x_n)$, and (iii) not-a-knot: $S'''$ continuous at $x_1$ and $x_{n-1}$.

Typically, one represents the individual cubic pieces as

$$s_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3, \quad j = 0, 1, \ldots, n$$

The equations above can be decoupled into an $n \times n$ tridiagonal linear system of equations in the unknowns $c_j$, and $3n$ equations in one unknown. The $a_j$'s are obvious from 1., and once the tridiagonal system is solved, the $b_j$'s and $d_j$'s are solved in terms of the $a_j$'s, $c_j$'s, $x_j$'s and $y_j$'s.