

## QR Iterations

Consider the iteration

$$\begin{aligned} Q_i R_i &\leftarrow A_i \\ A_{i+1} &\leftarrow R_i Q_i \end{aligned}$$

Here we have first computed the QR factorization of  $A_i$ , and then reversed their product to form  $A_{i+1}$ . From  $A_i = Q_i R_i$  we have  $R_i = Q_i^t A_i$ , and substituting that into  $A_{i+1} = R_i Q_i$  gives

$$A_{i+1} = Q_i^t A_i Q_i$$

and thus the QR step is a similarity transformation!

If the eigenvalues of  $A$  are all real, then this iteration almost always converges to an upper triangular matrix  $T$ . In this limit, the eigenvalues of  $T$  (and hence  $A$ , right?) are  $t_{11}, t_{22}, \dots, t_{nn}$ .  $T$  is called a *Schur form* for  $A$  and the eigenvectors of  $T$  are *Schur vectors* of  $A$ . Every matrix is unitarily similar to an upper triangular matrix, and  $T = Q^* A Q$  is called a *Schur decomposition* of  $A$ .

As it stands, this *QR iteration* requires  $O(n^3)$  flops per iteration. We can reduce this by an order of magnitude by first reducing  $A$  to Hessenberg form  $H_0 = Q^t A Q$ . The following iteration preserves the Hessenberg form, and if we use a Householder (or Givens) QR factorization it requires only  $O(n^2)$  flops:

$$\begin{aligned} Q_i R_i &\leftarrow H_i \\ H_{i+1} &\leftarrow R_i Q_i \end{aligned}$$

Notice that if a Hessenberg matrix  $H$  has  $h_{k+1,k} = 0$ , then the eigenproblem *decouples*: it is a block triangular matrix, and the eigenvalues of  $H$  are the union of the eigenvalues of the diagonal blocks (which are Hessenberg). A Hessenberg matrix for which none of the subdiagonal elements are zero is called *unreduced*.

Now if  $\lambda$  is an eigenvalue of an unreduced Hessenberg matrix  $H$ , then the QR factorization  $QR = H - \lambda I$  will have  $r_{nn} = 0$  (right?), and thus  $H_{new} = RQ + \lambda I$  will have last row  $\lambda e_n^t$ . So what?  $H_{new} = Q^t H Q$  is a reduced Hessenberg matrix: we just decoupled  $\lambda$ ! Now we don't usually know  $\lambda$  a priori, but we *can* speed convergence of the QR iterations by shifting  $H_i$  at each step by an approximate eigenvalue:

$$\begin{aligned} Q_i R_i &\leftarrow H_i - s_i I \\ H_{i+1} &\leftarrow R_i Q_i + s_i I \end{aligned}$$

This iteration is one of the most used methods to compute the eigenvalues and eigenvectors of symmetric (or Hermitian) matrices. In this case,  $H$  is both upper and lower Hessenberg, (called *tridiagonal*) and has only real eigenvalues. Furthermore, the QR iteration in this case requires only  $O(n)$  flops.

For nonsymmetric matrices we still have to address complex eigenvalues and the added cost complex arithmetic...