Interpolating Polynomials

Suppose we are given a set of $n+1$ points $(x_j, y_j), \quad j = 0, 1, \ldots, n$, with the x_j's distinct. Can we find a polynomial P which passes through, or interpolates, these points? Indeed, there are infinitely many. Let's look at what we are asking. Write P as

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m.$$

Notice that we have $m+1$ coefficients available to satisfy our $n+1$ interpolation conditions, so let's run with $m+1 = n+1$ and see what happens.

The interpolation conditions are $P(x_j) = y_j, \quad j = 0, 1, \ldots, n$, so for each j we get a linear equation in the a_j's:

$$a_0 + a_1 x_j + a_2 x_j^2 + \cdots + a_n x_j^n = y_j.$$

Writing these equations $n+1$ equations in matrix notation gives

$$Va = y,$$

where $a = (a_0, a_1, \ldots, a_n)^t$, $y = (y_0, y_1, \ldots, y_n)^t$, and $V = [x_i^j]_{i,j=0}^n$ is called a *Vandermonde* matrix. It is easy (but tedious) to show that the determininant of V is $\prod_{i>j}(x_i - x_j)$, so distinct nodes make V nonsingular. Therefore $Va = y$ has a unique solution representing the unique interpolating polynomial of degree $\leq n$.

There are many ways to represent this polynomial. We have just shown what it looks like in the standard ordered basis. There are a few other important polynomial bases for this problem. We will investigate the Lagrange basis; it is the most natural, as you will see. Let's define a set of polynomials

$$L_{nk}(x) = \prod_{i=0, i\neq k}^{n} \frac{x - x_i}{x_k - x_i}.$$

When restricted to the nodes, each of these polynomials is a delta function:

$$L_{nk}(x_j) = \delta_{jk} = \begin{cases} 0 & , \quad j \neq k \\ 1 & , \quad j = k \end{cases}$$

We call the $L_{nk}(x)$ Lagrange basis functions, and the representation

$$P(x) = \sum_{i=0}^{n} y_i L_{ni}(x)$$

the *Lagrange form* of the interpolator. You check that $P(x_j) = y_j, \; j = 0, 1, \ldots, n$.

Associated with the interpolating polynomial is the following error formula, which is valid for any function $f \in C^{(n+1)}([a,b])$, where $[a,b]$ is any interval containing the nodes.

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i), \qquad \xi \in (a,b).$$