Convergence of Newton's method

Newton's method uses a linear Taylor approximation to the function f to approximate its root. Let p be such that $f(p) = 0$, and let p_k be an approximation to p. Let us use the abbreviation $f_k \equiv f(p_k)$ throughout. If we take as our next approximation to p the root of the tangent line passing through (p_k, f_k) then we get Newton's method:

$$p_{k+1} = p_k - \frac{f(p_k)}{f'(p_k)}$$

We are interested here in analyzing the speed of convergence of Newton's method. To that end, define the error at the k^{th} step to be $e_k = p - p_k$. We assume f'' is continuous near p and use a Taylor approximation about p_k
$0 = f(p) = f(p_k + e_k) = f(p_k) + e_k f'(p_k) + e_k^2 f''(p_k)/2 + O(e_k^3)$. If $f'(p_k) \neq 0$, we may write

$$-\frac{f(p_k)}{f'(p_k)} = e_k + e_k^2 \frac{f''(p_k)}{2f'(p_k)} + O(e_k^3)$$

Then

$$\begin{aligned} e_{k+1} &= p - p_{k+1} = p - (p_k - \frac{f(p_k)}{f'(p_k)}) \\ &= e_k - (e_k + \tfrac{f''(p_k)}{2f'(p_k)}\, e_k^2 + O(e_k^3)) \\ &= -\tfrac{f''(p_k)}{2f'(p_k)}\, e_k^2 + O(e_k^3) \end{aligned}$$

We want to find α such that $|e_{k+1}| \to C|e_k|^\alpha$, so we take the limit, giving $\alpha = 2$ and $C = |f''(p)/(2f'(p))|$. When $f'(p) \neq 0$ we say that Newton's is (iteration) *quadratic*, since $\alpha = 2$.

In order to compare Newton's method to other methods, we must note that two functions need to be evaluated at each iteration of Newton's method: f and f'. We have no idea, in general, how much time it takes to evaluate f' relative to f; it could be easier or harder, or about the same. Unless we know more about f, we might as well just say that Newton's method requires 2 function evaluations per iteration. So what is the order of convergence of Newton's method relative to function evaluations? We use $|p_{n+2} - p| = C|p_n - p|^2$ (why?) to find α and λ in $|p_{n+1} - p| = \lambda|p_n - p|^\alpha$:

$$|p_{n+1} - p|^\alpha = \lambda^\alpha |p_n - p|^{\alpha^2},$$

and thus

$$|p_{n+2} - p| = \lambda|p_{n+1} - p|^\alpha = \lambda^{1+\alpha}|p_n - p|^{\alpha^2}.$$

This forces $\alpha^2 = 2$ and $\lambda^{1+\alpha} = C$, or $\lambda \approx |f''(p)/(2f'(p))|^{0.414}$. Therefore, when comparing Newton's method to other methods which require only one function evaluation per iteration, we should consider Newton's method to have *superlinear* convergence of order $\alpha = \sqrt{2}$, *not* the quadratic convergence of order $\alpha = 2$.