Osculating Polynomials

As you have probably guessed, there is a more general idea growing here. Suppose that at a given node x_j, we know a function value y_j and maybe several derivative values $y'_j,\ y''_j, \ldots, y_j^{(m_j)}$. We can use the Vandermonde matrix to show that there is a unique polynomial P of degree no more than d which satisfies

$$P^{(k)}(x_j) = y_j^{(k)}, \quad k = 0, 1, \ldots, m_j, \quad j = 0, 1, \ldots, n$$

where $d = n + m_0 + m_1 + \cdots + m_n$. This *osculating* (kissing) polynomial is a very general polynomial interpolator.

In fact, you should be able to name the type of polynomial approximation associated with each of the following types of data and give its degree:

1. $n > 0, \quad m_j = 0, \quad j = 0, 1, \ldots, n.$
2. $n > 0, \quad m_j = 1, \quad j = 0, 1, \ldots, n.$
3. $n = 0, \quad m_0 = N.$

The Vandermonde block associated with node j now has $1 + m_j$ rows, and looks like

$$\begin{bmatrix} 1 & x & x^2 & x^3 & \ldots & x^d \\ 0 & 1 & 2x & 3x^2 & \ldots & dx^{d-1} \\ 0 & 0 & 2 & 6x & \ldots & (d-1)dx^{d-2} \\ & & & & \vdots & \end{bmatrix}$$

evaluated at $x = x_j$. As you have guessed, the full $(d+1) \times (d+1)$ Vandermonde matrix is nonsingular iff the x_j are distinct.

Of course, in some applications the y_j are values of a known function f. If f is smooth enough, and $[a, b]$ contains all of the nodes, then $\forall x \in [a, b],\ \exists \xi \in [a, b]$ such that

$$f(x) = P(x) + \frac{f^{(d+1)}(\xi)}{(d+1)!} \prod_{j=0}^{n} (x - x_j)^{m_j+1}.$$

The use of one polynomial to interpolate a large set of points (or a large number of conditions on a set of points) requires that the interpolator have a large degree. This often gives large and unwanted oscillations in the interpolator, and makes evaluating $P(x)$ more costly. If one has the freedom to choose the nodes, they can be selected to minimize the wild oscillations; this is called Chebyshev node selection. But even in this case there may not be a polynomial interpolator of reasonable degree satisfying your constraints.