Rejected

Our first look at Monte-Carlo integration showed that when we could sample a uniform r.v. from our domain Ω, then

$$\text{volume}(\Omega) f_{ave} = \int_\Omega f(X) dV$$

could be used to approximate $\int_\Omega f(X)dV$ as

$$\int_\Omega f(X)dV = \text{volume}(\Omega)\bar{f} + \text{O}(1/\sqrt{N}) = S + \text{O}(1/\sqrt{N})$$

where $\bar{f} = (\sum_{i=1}^{N} f(x_i))/N$ is our computed sample mean.

Typically $\Omega \subset \mathbb{R}^d$ and d is large. If, e.g., Ω is a rectangular region, say $\Omega = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_d, b_d]$, then volume$(\Omega) = \prod_{j=1}^{d}(b_j - a_j)$, and sampling from Ω is a matter of creating a vector $x = (x_1, x_2, \ldots, x_n)$, where x_i is a uniform (pseudo-)random variable on $[a_i, b_i]$. Often it is a challenge to compute volume(Ω) and/or to sample from a uniform distribution on Ω. One technique that addresses both challenges is *rejection sampling*.

Consider a domain $D \subset \mathbb{R}^d$ for which (i) $\Omega \subset D$, (ii) volume(D) is known, and (iii) from which we can sample from a uniform distribution. If we sample $x \in D$, then either $x \in \Omega$ or not. The ratio of the number of points in Ω to those in D approaches the ratio volume(Ω)/volume(D). Furthermore, the sampling of those $x \in \Omega$ is from a uniform distribution *on* Ω. Therefore, if we sample M times from D and N of those are in Ω, then $\lim_{N\to\infty} N/M = \text{volume}(\Omega)/\text{volume}(D)$ and in fact

$$\int_\Omega f(X)dV = \left[\frac{N}{M}\text{volume}(D)\right]\left[\frac{\sum_{i=1}^{N} f(x)}{N}\right] \; + \; \text{O}(1/\sqrt{N}).$$

Here is the fundamental rejection sampling technique:

```
s = 0;  j = 0;  M = 0;
while j < N,
    sample x from uniform distribution on D
    if x is in Omega then
        j = j + 1
        s = s + f(x)
    end
    M = M + 1
end
S = s*volume(Omega)/M
```

The smaller $D - \Omega$, the fewer samples are rejected. Often the biggest computation in the loop is to decide if $x \in \Omega$, and this question must be answered for each $x \in D$.