Example: Sensitivity of Linear Systems

Consider the vectors

$$a_1 = \begin{pmatrix} 0.999 \\ 1 \end{pmatrix}, \text{ and } a_2 = \begin{pmatrix} 1 \\ 1.001 \end{pmatrix}.$$

Let's let the matrix A have columns a_1 and a_2:

$$A = \begin{bmatrix} a_1, & a_2 \end{bmatrix} = \begin{bmatrix} 0.999 & 1 \\ 1 & 1.001 \end{bmatrix}.$$

You can check that A is nonsingular. Then a_1 and a_2 are linearly independent, and so any $b \in \mathbb{R}^2$ can be written in exactly one way as a linear combination $b = x_1 a_1 + x_2 a_2$. The coefficients x_1 and x_2 of this combination are the coordinates of the solution $x = (x_1, x_2)^t$, of the matrix equation $Ax = b$.

Now let's take b to be

$$b = \begin{pmatrix} 1.9989 \\ 2.0010 \end{pmatrix}.$$

When we solve $Ax = b$, we find that (no rounding errors here)

$$x = \begin{pmatrix} 101.1 \\ -99 \end{pmatrix}.$$

that is,

$$b = 101.1 \ a_1 - 99 \ a_2,$$

Now suppose we round b to the nearest thousandths place:

$$\tilde{b} = \begin{pmatrix} 1.999 \\ 2.001 \end{pmatrix}.$$

This small change in b can be measured: $\|b - \tilde{b}\|_\infty = 0.0001$. Now how much of a_1 and a_2 do we need to make $\tilde{b}$? Well, we solve $Ax = \tilde{b}$ to get

$$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

that is,

$$\tilde{b} = 1 \ a_1 + 1 \ a_2.$$

A change in b of about 10^{-4} gives a change in x of over 10^2.

This example was designed to make a point, you can see how it works by interpreting $Ax = b$ as the intersection of 2 lines. Go ahead and plot the lines...