Stiff Differential Equations

Linear differential equations are tremendously important as models of many physical phenomena, and all (well posed) problems behave like linear equations locally. We should therefore expect that our numerical methods behave well when applied to linear problems.

Consider first the initial value problem

$$y'(t) = \lambda y(t), \quad y(0) = \alpha, \quad t \geq 0. \qquad \text{(test)}$$

It's separable, and we can find the solution $y(t) = \alpha e^{\lambda t}$. The case $\lambda \geq 0$ will be deferred (it's scarier). With $\lambda < 0$, $y(t) \to 0$ as $t \to \infty$ (and pretty quickly, too). If our method is to approximate even this simple test problem, we need $w_{k+1} \to 0$ as $k \to \infty$. Let's see where this leads...

Recall that the single step methods $w_{k+1} = w_k + h\phi(t_k, w_k)$, applied to (test) have the ideal slope $\phi^I(t, y) = (y(t+h) - y(t))/h = y(t)(e^{\lambda h} - 1)/h$. If we write $y(t+h) = Cy(t)$, then $C = e^{\lambda h}$, and we see that, when applied to (test), $w_k + h\phi(t_k, w_k)$ should approximate $e^{\lambda h} w_k$. How well it does this is says volumes about the method, but at the coarsest level, $w_{k+1} = Cw_k \to 0$ as $k \to \infty$, requires $|C| < 1$.

To see the implications, let's go back to Euler's method. Here $\phi(t, y) = f(t, y) = \lambda y$, and writing $w_{k+1} = Cw_k$ we have $C = 1 + h\lambda$. Now $|C| < 1$ means $h < 2/|\lambda|$, and we have another constraint on h. You should convince yourself that the Taylor method of order n has $C = \sum_{j=0}^{n} (\lambda h)^j / j!$. In fact the RK methods of order up to 4 are exactly Taylor methods when applied to (test). For $n = 4$, h must satisfy $h < 2.7853/|\lambda|$.

This may not seem like a terribly interesting constraint on h, for example, it does not typically force h to be so small that rounding errors swamp discretization errors. It does, though, impose an upper bound that can trick an adaptive method, even for moderate λ. To see this we usually talk about *stiffness* in the context of a *system* of IVP's. The test equation here is $y' = Ay$. Assuming all eigenvalues of A have negative real parts (i.e. $\Re(\lambda_i) < 0$), we measure stiffness as max $\frac{\Re(\lambda_i)}{\Re(\lambda_j)}$. If this were a coupled spring-mass system, this would be the ratio of the spring constants of the most and least stiff springs.

If this ratio is large, then the motion is an extreme mixture of high and low frequencies. The method must take care not to exceed the h-constraint for the highest frequency (most negative λ). Usually such systems are damped, and then the high frequency component of the solution is a transient event that once decayed leaves a smoother solution. This is all expected. So what? Here's the crux of the biscuit: to take advantage of the smoother solution we would like to take bigger time steps after the higher frequencies decay, but if h is choosen larger than the constraint *for the highest frequency*, it will get excited by errors and reappear!