# Runge Kutta Methods of Order Two

$$y'(t) = f(t, y), \quad t \in [a, b], \quad y(a) = \alpha \qquad \text{(IVP)}$$

So the Runge-Kutta methods are single step methods that give us smaller errors than Euler, and more generality than the Taylor methods. We know that they sample $f$ (the slope field) in the interval $[t, t+h]$ in order to approximate the average (*and ideal*) slope $(y(t+h) - y(t))/h$. *How* this is done is too much to cover in all of its (beautiful) generality, but we will explore the $2^{nd}$ order methods here.

Recall that the $2^{nd}$ order Taylor method is derived by dropping the $\text{O}(h^3)$ term from

$$\begin{aligned} y(t+h) &= y(t) + hy'(t) + \tfrac{h^2}{2} y''(t) + \text{O}(h^3) \\ &= y(t) + hf(t, y) + \tfrac{h^2}{2}[f_t(t, y) + f(t, y) f_y(t, y)] + \text{O}(h^3). \end{aligned}$$

The iteration looks like

$$w_{j+1} = w_j + hf(t_j, w_j) + \frac{h^2}{2}[f_t(t_j, w_j) + f(t_j, w_j) f_y(t_j, w_j)].$$

It is the $f_t$ and $f_y$ terms that restrict the general use of this method, so we will try to replace these. To that end we introduce the first order Taylor polynomial in two variables

$$f(t+h, y+k) = f(t, y) + hf_t(t, y) + kf_y(t, y) + \text{O}(h^2 + hk + k^2).$$

Our method will have the form

$$w_{j+1} = w_j + h[\lambda f(t_j, w_j) + (1-\lambda) f(t_j + \alpha h, w_j + \alpha h f(t_j, w_j)],$$

with $\alpha \in (0, 1]$. Matching $f(t_j + \alpha h, w_j + \alpha h f(t_j, w_j)$ to $f(t+h, y+k)$ gives $k = \alpha h f(t_j, w_j)$, and replacing $f(t_j + h, w_j + k)$ with the Taylor form gives

$$\begin{aligned} w_{j+1} &= w_j + h[\lambda f(t_j, w_j) + (1-\lambda)(f(t_j, w_j) + hf_t(t_j, w_j) + \alpha h f(t_j, w_j) f_y(t_j, w_j))] \\ &= w_j + hf(t_j, w_j) + h(1-\lambda)[\alpha h f_t(t_j, w_j) + \alpha h f(t_j, w_j) f_y(t_j, w_j))]. \end{aligned}$$

Comparing this to the Taylor iteration, we see that we must have

$$(1-\lambda)\alpha = \frac{1}{2}.$$

Here, then, is the general form for all second order Runge-Kutta methods:

$$w_{j+1} = w_j + \frac{h}{2\alpha}[\ (2\alpha - 1) f(t_j, w_j) + f(\ t_j + \alpha h,\ w_j + \alpha h f(t_j, w_j)\ )\ ].$$

Most authors include the formulas for $\alpha = \frac{1}{2}$ (the midpoint method), and $\alpha = 1$ (modified Euler), but in fact there are a continuum of methods for $\alpha \in [\frac{1}{2}, 1]$. If $\alpha \notin [\frac{1}{2}, 1]$, then $\lambda \notin [0, 1]$. $\alpha < \frac{1}{2}$ may be ok, but if $\alpha$ is too small, we do not even have an order $h^2$ method (since $k = \alpha h f(t_j, w_j)$), while if $\alpha > 1$, we are reaching outside of $[t, t+h]$ to average something in $[t, t+h]$.