Runge-Kutta Methods

$$y'(t) = f(t,y), \quad t \in [a,b], \quad y(a) = \alpha \qquad \text{(IVP)}$$

Fortunately, there are several ways to achieve the higher order l.t.e. of the Taylor methods without the need to evaluate $f', f'', \ldots$ The most popular single-step IVP solvers currently in use are the Runge-Kutta (R-K) methods.

Recall that $m = (y_{k+1} - y_k)/h$ is the ideal value for the quantity $\phi(t_k, y_k)$ in the single-step method $w_{k+1} = w_k + h\phi(t_k, w_k)$. Now view m as the average value of y' over the interval $[t_k, t_{k+1}]$. The R-K perspective is to approximate m by averaging approximate samples of y' from this interval. Euler's method is the R-K method which approximates the average value of the slope on $[t_k, t_{k+1}]$ by an approximation to its value at the left endpoint:

$$m \approx y'(t_k) \approx f(t_k, w_k) \equiv \phi_{\text{Euler}}.$$

If we think the left endpoint is not as representative as the midpoint, we can use

$$m \approx y'(t_k + h/2) \approx f(t_k + \tfrac{h}{2},\ w_k + \tfrac{h}{2} f(t_k, w_k)) \equiv \phi_{\text{Midpoint}}.$$

Maybe an average of the left and right endpoints seems better

$$m \approx \tfrac{1}{2}(y'(t_k) + y'(t_k + h)) \approx \tfrac{1}{2}(f(t_k, w_k) + f(t_k + h, w_k + hf(t_k, w_k))) \equiv \phi_{\text{ModEuler}}.$$

These last two (Midpoint and Modified Euler) methods have l.t.e. $O(h^2)$ and require 2 f-evals per iteration. R-K methods all have the form:

$$\phi_{\text{RK}} = \sum_{i=1}^{m} r_i f(t_k + \tau_i,\ K_i),$$

where $\tau_i \in [0, h]$ and K_i is another sum of f-values evaluated (hopefully) near the solution graph in $[t_k, t_{k+1}]$.

For now we avoid the question of *how* to construct the weighted average of nested function evaluations that give a method with a certain l.t.e., but in order to do that one needs to determine where to sample the slope field (which τ_i and K_i above), and what weights to give these values (which r_i).

There has been a tremendous amount of work done in this area, and we will end this too brief introduction by stating a theorem which tells us the best possible l.t.e. that can be achieved by a R-K method using n function evaluations per step.

Theorem:

# f-evals per step	2	3	4	$n = 5, 6, 7$	$n = 8, 9$	$n = 10, 11$
best possible l.t.e.	$O(h^2)$	$O(h^3)$	$O(h^4)$	$O(h^{n-1})$	$O(h^{n-2})$	$O(h^{n-3})$