Hermite Interpolation

Suppose again that we are given a set of points $(x_j, y_j),\quad j = 0, 1, \ldots, n$. We found that with the x_j's distinct (no conditions at all on the y_j's), there is a unique polynomial of degree no more than n that interpolates these points. Here we show that we can control the shape of the interpolator as much as we please.

We would like to have the ability to include other information in the interpolator. The most common generalization is to include derivative information. We can add to the original interpolation conditions $P(x_j) = y_j,\quad j = 0, 1, \ldots, n$, the derivative conditions

$$P'(x_j) = y'_j, \quad j = 0, 1, \ldots, n, \quad \text{or}$$
$$a_1 + 2a_2 x_j + \cdots + 2n a_{2n} x_j^{2n-1} + (2n+1) a_{2n+1} x_j^{2n} = y'_j.$$

Now for each node x_j, we need two numbers y_j and y'_j. This increase of input data allows us more influence on the interpolators shape, but requires that we about double its degree from $(n+1) - 1$ to $(2n+2) - 1$. For each j the two linear equations might correspond to two rows

$$\begin{bmatrix} 1 & x_j & x_j^2 & x_j^3 & \ldots & x_j^{2n+1} \\ 0 & 1 & 2x_j & 3x_j^2 & \ldots & (2n+1)x_j^{2n} \end{bmatrix}$$

of a *generalized Vandermonde system*, $Va = y$, where $a = (a_0, a_1, \ldots, a_{2n+1})^t$, and $y = (y_0, y'_0, y_1, y'_1, \ldots, y_n, y'_n)^t$. V is nonsingular iff the x_j are distinct, and thus there is a unique polynomial of degree no more than $2n+1$ which interpolates the data $(x_j, y_j),\ (x_j, y'_j)$; it is called the *Hermite interpolating polynomial.* There are explicit formulas for this polynomial in various bases, but they are simply different representations for the polynomial P above, whose coefficients are $a = V^{-1}y$.

The Lagrange form for the Hermite polynomial takes a very nice form for theoretical work. Define

$$H_{n,i}(x) = [1 - 2(x - x_i)L'_{n,i}(x_i)]L^2_{n,i}(x) \quad \text{and} \quad \hat{H}_{n,i}(x) = (x - x_i)L^2_{n,i}(x),$$

where the $L_{n,i}(x)$ are the standard Lagrange basis polynomials. Check out this handy set of nodal properties:

$$\begin{aligned} H_{n,i}(x_j) &= \delta_{ij}, \quad & \hat{H}_{n,i}(x_j) &= 0, \\ H'_{n,i}(x_j) &= 0, \quad & \hat{H}'_{n,i}(x_j) &= \delta_{ij}. \end{aligned}$$

It is easy to verify now that the "Lagrange form" of this Hermite interpolator is

$$P(x) = \sum_{j=0}^{n} [y_j H_{n,j}(x) + y'_j \hat{H}_{n,j}(x)].$$

If the data are associated with a smooth enough function, then we have an error formula: If $y_j = f(x_j)$, $y'_i = f'(x_j)$ and $[a, b]$ contains the nodes, then $\exists \xi \in [a, b]$ with

$$f(x) = P(x) + \frac{f^{(2n+2)}(\xi)}{(2n+2)!} \prod_{j=0}^{n} (x - x_j)^2.$$