Gaussian Quadrature

Recall the general quadrature rule

$$\int_a^b f(x)\ dx = \sum_{i=0}^{n} c_i f(x_i) + R,$$

where the *nodes* $x_0 < x_1 < \cdots < x_n \in [a,b]$ are arbitrary, $c_i = \int_a^b L_{ni}(x)\ dx$, $L_{ni}(x)$ is the i^{th} Lagrange basis function for the nodes, and R is the truncation error (see the page *Quadrature*).

Notice that if f is a polynomial of degree n or less, then R is 0, and the rule is *exact.* We say that the *order* of a quadrature rule is the highest degree polynomial for which the rule is exact. Thus the rule above has order at least n.

The *Newton-Cotes* rules use equally space nodes, but if we have freedom to choose our nodes we can do much better:

Let $x_0 < x_1 < \cdots < x_n \in [a,b]$ and let us find a polynomial $p_n(x) = c\prod_{i=0}^{n-1}(x - x_i)$ for which

$$\int_a^b x^k p_n(x)\ dx = 0, \quad k = 0, 1, \ldots, n-1 \qquad\qquad (\mathrm{O}).$$

Whether we can do this is a good question, but if we can, then behold: If h is any polynomial of degree $2n-1$ or less, then we can write $h = qp_n + r$, where q and r are polynomials of degree at most $n-1$, and

$$\int_a^b h(x)\ dx = \int_a^b q(x)p_n(x)\ dx + \int_a^b r(x)\ dx.$$

But $\int_a^b q(x)p_n(x)\ dx = 0$ by conditions (O), and $h(x_i) = r(x_i)$ because $p_n(x_i) = 0$. So

$$\int_a^b h(x)\ dx = \int_a^b r(x)\ dx = \sum_{i=0}^{n} c_i r(x_i) = \sum_{i=0}^{n} c_i h(x_i).$$

These conditions on $p_n(x)$ give a *Gaussian Quadrature* (GQ) rule, which is of order at least $2n-1$, nearly doubling the order of the Newton-Cotes rules!. Remarkable? You decide. Beautiful? Oh, man! Attainable? Glad you asked:

Notice that (O) are the orthogonolity conditions $<x^k, p_n(x)>_{[a,b]} \;=\; 0$, and thus we want n nodes, pairwise distinct in $[a,b]$, to give a p_n which is orthogonal (on $[a,b]$ wrt the weight $w(x) \equiv 1$) to all polynomials of degree less than n.

For *Legendre* GQ, we map $[a,b]$ to $[-1,1]$ ($u = 2(x-a)/(b-a) - 1$), and make use of the set of polynomials orthogonal on $[-1,1]$ wrt weight $w(x) \equiv 1$. That is, $p_n(x)$ is (some normalization of) the n^{th} Legendre polynomial (which satisfies (O)). It has the roots we want: Let $g(x) = \prod_{i=1}^{m}(x - x_i)$ be the monic polynomial of degree m with roots consisting of the roots of p_n of odd multiplicity in $(-1,1)$, then $|\int_{-1}^{1} g(x)p_n(x)\ dx| > 0$. If $m < n$, this integral is 0, so $m \geq n$, and thus p_n must have n distinct roots in $(-1,1)$. There are other intervals and weight functions which allow GQ for singular functions and infinitely wide intervals, but Legendre GQ is far and away the most used.