# The Francis Algorithm

Recall that we have a shifted QR iteration that converges quickly to a *reduced* Hessenberg matrix if the $s_i$ are close to an eigenvalue of $H$ (in fact, only one iteration does it if $s_i$ is an eigenvalue of $H$ (this is called an *ultimate shift*)). If $H$ is reduced, we can decouple (or deflate) and continue with a strictly smaller problem than before. Here is the iteration:

$$\begin{aligned} Q_i R_i &= H_i - s_i I \\ H_{i+1} &= R_i Q_i + s_i I = Q_i^t H_i Q_i \end{aligned}$$

Nonsymmetric real matrices may have complex eigenvalues, which must occur in conjugate pairs: $u + iv$ and $u - iv$. If we apply a complex shift $s_i = u + iv$ to the iteration, then $Q_i$, $R_i$ and $H_{i+1}$ will be complex. This requires more storage, and more computation (one complex multiplication requires 4 real multiplications and 2 real additions).

Now if we immediately follow the $s_i = u + iv$ shift iteration with an iteration with shift $s_{i+1} = u - iv$, then everything becomes again real. We can understand this by noting that two iterations above gives the same $H_{i+2}$ as the single iteration

$$\begin{aligned} QR &= (H_i - s_i I)(H_i - s_{i+1} I) \\ H_{i+2} &= Q^t H_i Q. \end{aligned}$$

Even more: $R = R_{i+1} R_i$ and $Q = Q_i Q_{i+1}$. So here is a way to apply two complex conjugate shifts in succession using only real arithmetic! While it appears, that we have bought this efficiency at the cost of forming $G = (H_i - s_i I)(H_i - s_{i+1} I)$, a wonderful uniqueness result comes to our aid:

The *Implicit Q Theorem* says that if $H_{i+2} = Q^t H_i Q$ is unreduced, then it is essentially uniquely determined by $H_i$ and the first column of $Q$.

Thus we need only compute enough of $G$ to determine the first column of $Q$, the remainder of $Q$ is discovered through a Hessenberg reduction. Here are the details:

$$QRe_1 = Ge_1 \implies Qe_1 = \pm Ge_1 / \|G\|_2.$$

So define a Householder reflector, $P_0$, so that $P_0 G e_1 = \alpha e_1$ (the first step of the Householder QR factorization of $G$), and apply this *not* to $G$, but to $H_i$ as a similarity transform: $B = P_0 H_i P_0$. $B$ is no longer Hessenberg, it has a bulge at $b_{41} \neq 0$, so we need reflectors $P_1, P_2, \ldots, P_{n-2}$ to "chase the bulge" from $b_{41}$ to $b_{52}$ to ... to $b_{n,n-2}$, resp. to "re-Hessenbergize" $H_i$:

$$H_{i+2} = P_{n-2} \cdots P_1 P_0 H_i P_0 P_1 \cdots P_{n-2} = Q^t H_i Q.$$

All of this was developed in J. Francis' 1961 paper, along with a scheme for choosing the shifts $s_i$ and $s_{i+1}$ as the eigenvalues of the lower right $2 \times 2$ submatrix of $H_i$. While fallable, this method – augmented with schemes for detecting subdiagonal elements "small enough" to allow decoupling and deflation – is the state of the art general purpose eigenvalue method.