

## The Action of Householder Reflectors

Let  $A \in \mathbb{R}^{m \times n}$ ,  $m \geq n$ . You may recall that the Householder QR factorization can be written as

$$H_p \cdots H_2 H_1 A = R,$$

where  $p = \min(n, m - 1)$  and  $H_k = I - (\frac{2}{u_k^t u_k}) u_k u_k^t$  is a Householder reflector which introduces zeros into positions  $k + 1$  to  $m$  of the  $k^{\text{th}}$  column of the matrix  $A^{(k-1)} = H_{k-1} \cdots H_2 H_1 A$ . This gives  $A = QR$ , where we define

$$Q = H_1 H_2 \cdots H_p.$$

The  $H_j$  are not explicitly formed, since that is very inefficient, and since they are completely determined by the  $u$  vectors. The matrix  $Q$  is rarely formed, either. What we usually do is save the  $u_j$ 's. If we let  $Q_u$  be the array whose  $j^{\text{th}}$  column is  $u_j$ , it would be called the "factored form" of  $Q$ . In Matlab, if  $u$  is our variable name for  $u_j$ , then we might write  $Qu(:, j) = u$ . In a memory efficient code, we would only save entries  $j : m$  of  $u_j$  ( $\tilde{u}_j = u_j(j : m)$ ), and would use the lower triangle of  $A$  (plus an  $n$ -vector) to store  $\tilde{u}_j$ ,  $j = 1, \dots, p$ .

Later we will need to find the vector  $c = Q^t b$ , and instead of explicitly forming  $Q$ , we use the routine HOUSEQACT. Notice that

$$C = Q^t B = H_p \cdots H_2 H_1 B,$$

so we write HOUSEQACT to form this product:

```
C = B;
for j=1:p,
    C = Hj * C; %but you code this line the right way...
end
```

Of course, it would be just as easy to write code to implement

$$C = QB = H_1 H_2 \cdots H_p B,$$

and  $BQ^t$  or  $BQ$ . In fact, while it is rarely needed explicitly, we could compute  $Q$  above as  $Q = QI$ , or  $Q = IQ, \dots$

We discuss how to find the  $u_j$ 's and how to code the algorithm above efficiently for speed and memory on other pages, but a good implementation of the loop above should be all the convincing you need to understand that Householder reflectors are never explicitly computed. If  $C \in \mathbb{R}^{m \times r}$ , a good implementation of the pseudocode above requires  $r \sum_{j=0}^n 4(m - j) \approx 2mnr(2 - \frac{n}{m})$  flops, and even if we knew  $Q \in \mathbb{R}^{m \times m}$  explicitly, that matrix product would require  $2m^2r$  flops (which is more costly than HOUSEQACT since  $n \leq m$ ).