Sensitivity of Simple Eigenvalues

Let $A \in \mathbb{C}^{n\times n}$. We would like to know how small perturbations in A change its eigenvalues. Of course $\partial\lambda_k/\partial a_{ij}$ measures just this sensitivity, but it isn't practical to compute these n^3 quantities. Suppose $Ax = \lambda x$, $y^*A = \lambda y^*$, and $\|x\|_2 = 1 = \|y\|_2$ (so x and y^* are respectively right and left eigenvectors associated with a simple (not multple) eigenvalue λ of A). Remember, $y^* = \bar{y}^t$ is a conjugate transpose.

Consider the perturbed eigenpair $x(t), \lambda(t)$ of the matrix $A + tE$:

$$(A + tE)x(t) = \lambda(t)x(t),$$

where $x(0) = x$, $\lambda(0) = \lambda$, and $\|x(t)\|_2 = 1$. Differentiating wrt t gives:

$$(A + tE)\dot{x}(t) + Ex(t) = \lambda\dot{x}(t) + \dot{\lambda}x(t).$$

Premultiplying by y^* gives $\dot{\lambda}(0) = y^*Ex/(y^*x)$, and for $\lambda \neq 0$, Taylor's theorem says

$$\begin{aligned} \left|\tfrac{\lambda(t)-\lambda}{\lambda}\right| &= \left|\tfrac{t\dot{\lambda}}{\lambda} + \mathrm{O}(t^2)\right| \\ &\approx \left|\tfrac{ty^*Ex}{\lambda y^*x}\right| \\ &\leq \tfrac{\|A\|}{|\lambda y^*x|}\tfrac{\|tE\|}{\|A\|}. \end{aligned}$$

Thus we say the relative (and absolute) condition numbers for λ are

$$\kappa(\lambda) = \frac{\|A\|}{|\lambda|}\frac{1}{|y^*x|} \qquad (\text{ and } \qquad \nu(\lambda) = \frac{1}{|y^*x|}).$$

So, if y^*x is small, λ is illconditioned. If λ is simple, y^*x cannot be zero, but for some matrices it can be very small. Let's take a diagonalizable matrix under consideration. Suppose $X^{-1}AX = \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. The columns of X are (normalized) right eigenvectors of A, while the rows of $X^{-1} = DY$ are (un-normalized) left eigenvectors. Let $D = \mathrm{diag}(\|e_i^tX^{-1}\|,\ i = 1, \ldots, n)$, so that the rows of Y have unit length. Then from above,

$$\nu(\lambda_i) = |e_i^tYXe_i|^{-1} = d_{ii}.$$

Therefore, if $\|X\|\|X^{-1}\|$ is large, then at least one eigenvalue of A is illconditioned. Conversely, if $\|X\|\|X^{-1}\|$ is not large, then all eigenvalues of A are well conditioned. Some eigenvalues of A may be sensitive, while others may not, but to describe the absolute sensitivity of the eigenvalues of A with a single number, a good choice is

$$\nu_{eig}(A) = \kappa_{inv}(X) = \|X\|\|X^{-1}\|.$$

A matrix A is called *normal* if $AA^* = A^*A$. Hermitian matrices are normal, as are unitary and diagonal matrices. The eigenvectors of normal matrices can always be arranged to be orthonormal, so if A is normal, then $\nu_{eig}(A) = 1$. The farther from normal a matrix is, the more sensitive (illconditioned) are its eigenvalues.