Crank-Nicolson

$$u_t = u_{xx}, \quad x \in (0,1), \quad t \in (0,T],$$

$$u(x,0) = f(x), \quad x \in (0,1),$$

$$u(0,t) = g(t), \quad t \in (0,T], \qquad u(1,t) = h(t), \quad t \in (0,T]$$

Here we have a 2^{nd} order linear homogeneous parabolic pde, with initial conditions, f, and (Dirichlet) boundary conditions, g and h. It is often called the *heat equation* or *diffusion equation*, and we will use it to discuss numerical methods which can be used for it and for more general parabolic problems.

Finite difference methods replace the differential operators (here $\frac{\partial}{\partial t}$ and $\frac{\partial^2}{\partial x^2}$) with difference operators (numerical differentiation formulae). For example, we can approximate $u_t(x,t)$ by $[u(x,t+k) - u(x,t)]/k$, and can replace the term u_t in the equation with the term $(U_{i,j+1} - U_{ij})/k$. This replacement introduces an $\mathrm{O}(k)$ truncation error. If we now replace $u_{xx}(x,t)$ with $[U_{i+1,j} - 2U_{ij} + U_{i-1,j}]/h^2$, then on a grid with time increment k and space increment h, $u_t = u_{xx}$ becomes

$$(U_{i,j+1} - U_{ij})/k = (U_{i+1,j} - 2U_{ij} + U_{i-1,j})/h^2$$

which models the heat equation with truncation error $\mathrm{O}(k + h^2)$. This can be solved for $U_{i,j+1}$, and letting $r = k/h^2$, gives the equation

$$U_{i,j+1} = rU_{i+1,j} + (1-2r)U_{ij} + rU_{i-1,j}.$$

Implemented as a single-step method this has l.t.e. $\mathrm{O}(k^2 + kh^2)$ for $r \in (0, \frac{1}{2}]$.

We will return here in another page to analyze these truncation errors, but let's push on. If $r > \frac{1}{2}$, then the method is unstable and this forces a very small time step $k = \mathrm{O}(h^2)$. Notice that using a backward difference formula for u_t gives

$$(U_{i,j+1} - U_{ij})/k = (U_{i+1,j+1} - 2U_{ij+1} + U_{i-1,j+1})/h^2,$$

and averaging these forward and backward difference equations gives

$$U_{i,j+1} - U_{ij} = r\ (\ \lambda[U_{i+1,j+1} - 2U_{ij+1} + U_{i-1,j+1}]\ +\ (1-\lambda)[U_{i+1,j} - 2U_{ij} + U_{i-1,j}]\ ).$$

Now we have 3 parameters at our disposal, h, k and λ. With $\lambda = 0$, we have the explicit method above, $\lambda = \frac{1}{2}$ gives the *Crank-Nicolson method*, and $\lambda = 1$ is called the *fully implicit* or the O'Brien form. This method is stable for all positive r as long as $\lambda \geq \frac{1}{2}(1 - \frac{1}{2r})$. For all positive λ, we need to solve a system of linear equations at each time step. In matrix form, the iteration looks like $A\mathbf{U}_{j+1} = B\mathbf{U}_j + b$, where b includes boundary conditions, and A and B are tridiagonal matrices. The ith interior rows of A and B are zero except

$$[a_{i-1,i} \quad a_{ii} \quad a_{i+1,i}] = [-r\lambda \quad (1+2r\lambda) \quad -r\lambda] \quad \text{and} \quad [b_{i-1,i} \quad b_{ii} \quad b_{i+1,i}] = [r(1-\lambda) \quad (1-2r(1-\lambda)) \quad r(1-\lambda)].$$