Composite Newton-Cotes

Recall that the Newton-Cotes rules were based on Lagrange interpolation, and that high degree polynomial interpolation typically generates wildly oscillating interpolants. This problem for the Newton-Cotes rules leads us naturally to the most often used rules for fixed data (e.g. equally spaced) quadrature.

Observe that for any $c \in [a, b]$ for which f is defined,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,$$

so we can construct rules over $[a, b]$ by piecing-together rules over subintervals of $[a, b]$. For example, the trapezoidal rule applied twice over $[a, b]$ gives

$$\begin{aligned} \textstyle\int_a^b f(x)dx &\approx \frac{\frac{a+b}{2}-a}{2}(f(a) + f(\tfrac{a+b}{2})) \quad + \quad \frac{b-\frac{a+b}{2}}{2}(f(\tfrac{a+b}{2} + f(b)) \\ &= \tfrac{h}{2}(f(a) + 2f(\tfrac{a+b}{2}) + f(b)). \end{aligned}$$

On $n + 1$ nodes and with $h = (b - a)/n$, the *composite trapezoidal rule* is

$$\int_a^b f(x)dx = \frac{h}{2}[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)] \quad - \quad \frac{b-a}{12}h^2 f''(\mu_1),$$

and (taking n even) the (very popular) *composite Simpson's rule* is

$$\int_a^b f(x)dx = \frac{h}{3}[f(x_0) + 2\sum_{i=1}^{n/2-1} f(x_{2i}) + 4\sum_{i=1}^{n/2} f(x_{2i-1}) + f(x_n)] \quad - \quad \frac{b-a}{180}h^4 f^{(4)}(\mu_2).$$

Hopefully you are wondering whether numerical integration is as illcondioned as numerical differentiation. Let's look at the composite Simpson's rule. As before we will model the rounding errors associated with the evaluation of f *and* the evaluation of the quadrature rule using $e(x)$ in $f(x) = \overline{f(x)} + e(x)$. We compute

$$\bar{I} \equiv \frac{h}{3}[\overline{f(x_0)} + \sum_{i=1}^{n/2} \overline{f(x_{2i})} + 4\sum_{i=1}^{n/2} \overline{f(x_{2i-1})} + \overline{f(x_n)}].$$

We then have

$$\int_a^b f(x)dx \quad = \quad \bar{I} \quad + \quad \frac{h}{3}[e(x_0) + 2\sum_{i=1}^{n/2} e(x_{2i}) + 4\sum_{i=1}^{n/2} e(x_{2i-1}) + e(x_n)] \quad - \quad \frac{b-a}{180}h^4 f^{(4)}(\mu_2),$$

where the 2^{nd} and 3^{rd} terms are the rounding and truncation error terms respectively. The rounding error term is actually a quadrature rule for $\int_a^b e(x)dx$, so we expect it to be small (why?), making the following a very pessimistic analysis. Let $|e(x)| \leq M_r$ on $[a, b]$. Then

$$\begin{aligned} |\textstyle\int_a^b f(x)dx - \bar{I}| &= \tfrac{h}{3}[e(x_0) + 2\textstyle\sum_{i=1}^{n/2} e(x_{2i}) + 4\sum_{i=1}^{n/2} e(x_{2i-1}) + e(x_n)] \quad - \quad \tfrac{b-a}{180}h^4 f^{(4)}(\mu_2) \\ &\leq hnM_r \quad - \quad \tfrac{b-a}{180}h^4 f^{(4)}(\mu_2) \\ &= (b-a)M_r \quad - \quad \tfrac{b-a}{180}h^4 f^{(4)}(\mu_2). \end{aligned}$$

In numerical differentiation, the error was unbounded as $h \to 0$. This time, as $h \to 0$, even this pessimistic error bound goes to $(b - a)M_r$, where (as before) M_r depends on the machine precision and the conditioning of "evaluate f at the nodes". Remarkably, if f' is bounded, the central limit theorem suggests that the total error behaves as μ $\mathrm{O}(\sqrt{h})$ as $h \to 0$.