

The residual vector for $Ax = b$

Suppose $A \in \mathbb{R}^{n \times n}$ is nonsingular, so that $x = A^{-1}b$ is the unique solution to $Ax = b$ and x solves $Ax = b$ if and only if the residual vector, $r = b - Ax$, satisfies $r = 0$.

Let \bar{x} be a computed approximation to x , and define

$$\bar{r} = b - A\bar{x}.$$

A measure (in units of $\|b\|$) of how much \bar{x} fails to satisfy $Ax = b$ is simply

$$\rho(\bar{x}) = \frac{\|\bar{r}\|}{\|b\|}. \quad (1)$$

This number, sometimes called the *relative residual*, might be the quantity you are interested in, but often we care about how well \bar{x} approximates the true solution x . It is important to note here that \bar{r} and ρ are quantities that we can compute from A , b , and \bar{x} , but x is forever unknown. We investigate below the relationship between $\rho(\bar{x})$ and both the actual error and the *backward error*.

We only require that the norm(s) being used are consistent, i.e. $\|Ax\| \leq \|A\|\|x\|$ for any $x \in \mathbb{R}^n$. We call $\kappa(A) \equiv \|A\|\|A^{-1}\|$ the condition number of A (more specifically, it is a relative condition number for the problem “given A , find A^{-1} ”).

Notice that $A^{-1}\bar{r} = A^{-1}b - A^{-1}A\bar{x}$, so

$$x - \bar{x} = A^{-1}\bar{r}.$$

From this we easily get the (relative) error bound

$$\epsilon(\bar{x}) \equiv \frac{\|x - \bar{x}\|}{\|x\|} = \frac{\|A^{-1}\bar{r}\|}{\|x\|} \leq \frac{\|A^{-1}\|\|\bar{r}\|}{\|x\|} \leq \frac{\|A^{-1}\|\|A\|\|\bar{r}\|}{\|b\|} = \kappa(A) \frac{\|\bar{r}\|}{\|b\|}.$$

From the other side:

$$\frac{1}{\kappa(A)} \frac{\|\bar{r}\|}{\|b\|} = \frac{\|b - A\bar{x}\|}{\|A\|\|A^{-1}\|\|b\|} \leq \frac{\|A(A^{-1}b - \bar{x})\|}{\|A\|\|x\|} \leq \frac{\|x - \bar{x}\|}{\|x\|} = \epsilon(\bar{x}),$$

together giving the tidy result

$$\frac{\rho(\bar{x})}{\kappa(A)} \leq \epsilon(\bar{x}) \leq \kappa(A)\rho(\bar{x}). \quad (2)$$

Finally we discuss the practical idea of backward error: Does \bar{x} solve a system close to $Ax = b$? More specifically, what's the smallest change we need to make to $Ax = b$ so that \bar{x} is a solution? Define the (relative) backward error in \bar{x} as

$$\beta(\bar{x}) \equiv \min_{\delta A, \delta b} \{ \|\delta A\|/\|A\| + \|\delta b\|/\|b\| \}, \quad \text{subject to } (A + \delta A)\bar{x} = b + \delta b.$$

Now notice that \bar{x} exactly satisfies the linear system $Ax = b - \bar{r}$ (this is just the definition of \bar{r}). So taking $\delta A = 0$ and $\delta b = -\bar{r}$, we know that $\beta(\bar{x}) \leq \|\bar{r}\|/\|b\|$ (this is because $(\delta A, \delta b) = (0, -\bar{r})$ is a feasible point for the minimization). Therefore,

$$\beta(\bar{x}) \leq \rho(\bar{x}). \quad (3)$$

Take these numbered equations with you...