

Projections

With the inner product $\langle x, y \rangle$, we have angles ($\langle x, y \rangle = \|x\|_2 \|y\|_2 \cos(\theta)$), and can speak of orthogonality: $x \perp y \iff \langle x, y \rangle = 0$. Here we will consider the standard inner product for \mathbb{R}^n : $\langle x, y \rangle \equiv x^T y$, but more general inner products can be very useful in many applications and algorithm development.

If S is a subspace of \mathbb{R}^n (write $S \leq \mathbb{R}^n$), we say $x \perp S$ if x is orthogonal to every element of S . The subspace $S^\perp \leq \mathbb{R}^n$ is called the *orthogonal complement* of S

$$S^\perp \equiv \{x \in \mathbb{R}^n : x \perp S\},$$

and $\mathbb{R}^n = S \oplus S^\perp$ is a direct sum decomposition of \mathbb{R}^n into complementary subspaces in such a way that each $x \in \mathbb{R}^n$ has the unique factorization $x = u + v$, with $u \in S$ and $v \in S^\perp$. In this setting, u is the *orthogonal projection* of x onto S . Complementary subspaces for any inner product space, V , like S and S^\perp , thus act very much like a basis of subspaces.

In this note, we will be looking at a transformation P , which satisfies

$$\forall x \in \mathbb{R}^n, Px = u, \text{ the orthogonal projection of } x \text{ onto } S.$$

If $x \in S$, then we should have $Px = x$ (right?), so P should satisfy $P^2 = P$, with $\text{Range}(P) = S$. The requirement that $Px = u \perp v$, with $u \in S$ and $v \in S^\perp$, combined with $x^T y = y^T x$ forces P to be self-adjoint (in matrix language over \mathbb{R} , $P^T = P$). Any linear transformation P which satisfies

1. $\text{Range}(P) = S$,
2. $P^2 = P$, and
3. $P^T = P$

is called an *orthogonal projector* onto S . If Q is another orthogonal projector onto S , then $Qx = Q(u + v) = u = Px$, $\forall x \in \mathbb{R}^n$, and hence $Q = P$ and we see that the orthogonal projector onto a subspace is unique. If we have a basis, then we should expect to be able to find a matrix representation for P ; call it P . Let X have linearly independent columns spanning S . Now (scratch paper handy?)

$PX = X \Rightarrow X^T P X = X^T X$, $\text{Range}(P) = \text{Span}(X) \Rightarrow P = X M$ for some M , $P = P^T \Rightarrow X M = M^T X^T$, $P = P^2 \Rightarrow P = X M M^T X^T$, giving $M M^T = (X^T X)^{-1}$ so that

$$P = X(X^T X)^{-1} X^T.$$

$X^T X$ is nonsingular, so the derivation and formula are perfectly reasonable, but if the columns of V form an *orthonormal basis* for S , then $V^T V = I$, in which case

$$P = V V^T.$$

Now from $x = Px \oplus v$, we see that $v = (I - P)x \in S^\perp$, and (check the properties) the orthogonal projector for S^\perp is $I - P$.