Gaussian Elimination as a Matrix Factorization

Each of the elementary row operations from Gaussian Elimination (GE) has associated with it a nonsingular matrix with the property that multiplying (on the left) by that associated matrix gives the same result as applying the row operation.

1. **Row operation 1** (R1): Multiply row i by a scalar $\alpha \neq 0$.
   Associated matrix: Let D be the identity matrix except for the (i,i) element, which is α. Then the matrix DA is the result of R1 applied to A.

2. **Row operation 2** (R2): Interchange row i and row j.
   Associated matrix: Let P be the identity matrix with rows i and j interchanged. Then the matrix PA is the result of R2 applied to A.

3. **Row operation 3** (R3): Multiply row j by a scalar m and add it to row i.
   Associated matrix: Let M be the identity but with m replacing the 0 in the (i,j) position. Then $M = I + me_ie_j^T$, and the matrix MA is the result of R3 applied to A. Taking $m = m_{ij} = -a_{ij}/a_{jj}$ puts a zero in the (i,j) position of MA.

Example Let $A = \begin{bmatrix} a_1^T \\ a_2^T \\ a_3^T \end{bmatrix} \in \mathbb{R}^{3\times n}$, that is, the i^{th} row of A is a_i^T.

Let $\quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad$ and $\quad M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ m & 0 & 1 \end{bmatrix}.$

Then $\quad DA = \begin{bmatrix} a_1^T \\ \alpha a_2^T \\ a_3^T \end{bmatrix}, \quad PA = \begin{bmatrix} a_1^T \\ a_3^T \\ a_2^T \end{bmatrix}, \quad$ and $\quad MA = \begin{bmatrix} a_1^T \\ a_2^T \\ a_3^T + ma_1^T \end{bmatrix}.$

A nonsingular matrix can *always* be triangularized by GE using only operations R2 and R3, and *almost always* using only R3:

If $a_{11} \neq 0$, then R3 can be used $n-1$ times to zero out all elements below the $(1,1)$ position. The new matrix, call it $A^{(1)}$, is related to A by

$$A^{(1)} = M_{n,1}M_{n-1,1}\cdots M_{21}A \equiv M_1A.$$

M_1 is called a *Gauss Transform*; take some time to digest this:

$$M_1 = M_{n,1}\cdots M_{21} = (I + m_{n,1}e_ne_1^T)\cdots(I + m_{21}e_2e_1^T) = I + m_1e_1^T,$$

where $m = (0, m_{21}, m_{31}, \ldots, m_{n,1})^T$. If $a_{kk}^{(k-1)} \neq 0, \quad k = 1, 2, \ldots, n-1$, then we can triangularize A:

$$A^{(k)} = M_{n,k}M_{n-1,k}\cdots M_{k+1,k}A^{(k-1)} \equiv M_k\cdots M_2M_1A,$$

and $A^{(n-1)} = M_{n-1}\cdots M_2M_1A \equiv U$ is upper triangular. Set $L = (M_k\cdots M_2M_1)^{-1}$. As a product of lower triangular matrices ($(I + m_ke_k^T)(I - m_ke_k^T) = I!$), L is lower triangular, and we have the factorization $A = LU$.