$A = LDM^T$

If A is nonsingular and $A = LU$, then we can set $D = \text{diag}(U)$ and (since D is nonsinglar) $M^T \equiv D^{-1}U$ is a unit upper triangular matrix and $A = LDM^T$.

There is no inherent benefit to this factorization over LU, but it can give us a perspective from which to develop other algorithms. The idea is not to compute LDM^T from LU, but to derive a method to compute L, D and M directly. To that end, consider the k^{th} column of $A = LDM^T$:

$$a \equiv Ae_k = LDM^T e_k \equiv Ly. \tag{1}$$

Suppose we have already know the first $k-1$ columns of L and consider the blocked treatment of the unit lower triangular system $a = Ly$:

$$\begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \tag{2}$$

where L_{11} is $k \times k$ and *known*, but the last column of L_{21} is something we would like to compute. Forward substitution gives y_1 as the solution to $L_{11}y_1 = a_1$. Now we know the first k elements of y, and (from (1))

$$y = DM^T e_k, \tag{3}$$

giving $e_k^T y = e_k^T DM^T e_k = d_{kk} e_k^T M^T e_k = d_{kk}$ (right?). $D^{-1}y = M^T e_k$ with M^T unit upper triangular means we can now compute the k^{th} column of M^T:
$e_k^T M = [y_1^T D_1^{-1},\ 0]$, where $D_1 = \text{diag}(d_{11}, d_{22}, \ldots, d_{kk})$.

Now let's try to find z, the k^{th} column of $L_{21} = [\tilde{L}_{21},\ z]$. Notice from (3) that M^T upper triangular means $y_2 = 0$, so $L_{21}y_1 + L_{22}y_2 = a_2$ from (2) reduces to $L_{21}y_1 = a_2$, or (with $y_1 = (\tilde{y}_1^T,\ d_{kk})^T$)

$$\tilde{L}_{21}\tilde{y}_1 + z d_{kk} = a_2,$$

giving $z = (a_2 - \tilde{L}_{21}\tilde{y}_1)/d_{kk}$.

Let's recap: Given the first $k-1$ columns of L and D, we can compute the k^{th} column of L, D and M^T as follows:

1. Solve $L_{11}y_1 = a_1$ for y_1 (forward substitution, about k^2 flops).
2. Set $d_{kk} = e_k^T y_1$.
3. Compute k^{th} row of M: $m_{kj} = e_j^T y_1/d_{jj},\ j = 1, 2, \ldots, k-1$ ($k-1$ flops).
4. Compute k^{th} column of L: $z = (a_2 - \tilde{L}_{21}\tilde{y}_1)/d_{kk}$ (about $2k(n-k)$ flops).

Like the LU factorization, this takes about $2n^3/3$ flops. And like the LU factorization, the method is not a stable general purpose method (trouble if d_{kk} is |small|), but can be stabilized with little extra effort.

Unlike the LU factorization, this method can take advantage of symmetry. If $A = A^T$, then $M = L$ and the $A = LDL^T$ factorization can be computed in $n^3/3$ flops, since step 4. can be skipped (it is done in step 3.).