Don't make it complicated

Let $A \in \mathbb{R}^{m \times n}$ have columns $a_1, a_2, \ldots, a_n$. To solve $Ax = b$ is simply to write b as a linear combination of the columns of A:

$$b = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n. \qquad (*)$$

A solution to $Ax = b$ is then the vector $x = (x_1, x_2, \ldots, x_n)^T$.

The subspace of $\mathbb{R}^m$ consisting of all linear combinations of the columns of A is a vector space called the range of A, or $\mathrm{ColSp}(A)$.

Is there a solution to $Ax = b$? Iff $b \in \mathrm{ColSp}(A)$.

If there is a solution, is that solution unique? Iff the columns of A are linearly independent.

Is there a unique solution? Iff $b \in \mathrm{ColSp}(A)$ *and* the columns of A are linearly independent.

Lame analogy: Our only tool is the 'linear combination', and our only materials are the columns of A. The set of things we can make is a vector space called the column space of A, our 'catalog'. Do we need all of our materials to make everything in our catalog, or is there redundancy? If removing a material doesn't shrink our catalog, then it is redundant. If we remove all redundancy we are left with a smallest set of materials with which we can construct our entire catalog. That set of materials is both necessary and sufficient (for our catalog), and is called a basis.

Don't miss the obvious:
The column space of any matrix, say A, is a vector space. The columns of A span that vector space. $Ax = b$ has a solution if and only if b is in that vector space. If the columns of A are linearly independent, then they form a basis for that vector space (the column space); in that case, $Ax = b$ has at most one solution.

There are many ways to pose these ideas, but basically: *consistent* is about 'span' (is b in our catalog?) and *unique* is about 'linear independence' (no redundant materials). Basis just combines these.