Inner Products give Geometry

The dot product is an example of an inner product. If x and y are two real 3-vectors, then $x \cdot y = x_1y_1 + x_2y_2 + x_3y_3$. If we think of x and y as (column) vectors in $\mathbb{R}^n$, then $x \cdot y = x^T y = \sum_{i=1}^n x_i y_i$ as a matrix multiplication. If x and y are (column) vectors in $\mathbb{C}^n$, then $x^T y$ is not well-behaved. We might instead use $\overline{x}^T y$ or $x^T \overline{y}$. These are both perfectly reasonable, equally well-behaved generalizations of the dot product to $\mathbb{C}^n$. Before you choose one, I should warn you that there are infinitely many perfectly reasonable generalizations of the dot product. We call them *inner products*.

An inner product is a function that takes two vectors and gives a scalar and which satisfies some properties that makes it "well-behaved". Specifically, if V is a vector space over the field $\mathbb{F}$, then

$$f : V \times V \to \mathbb{F}$$

is an inner product on $(V, \mathbb{F})$ if for all $x, y, z \in V$ and all $\alpha \in \mathbb{F}$

1. $f(x, x) > 0$ for all $x \neq 0$,
2. $f(x, y) = \overline{f(y, x)}$, and
3. $f(\alpha x + y, z) = \alpha f(x, z) + f(y, z)$.

When f is an inner product, we usually denote $f(x, y)$ by $<x, y>$. In this notation, the dot product on $\mathbb{R}^n$ is $x \cdot y = <x, y> = y^T x$, while the standard inner product on $\mathbb{C}^n$ is $<x, y> = \overline{y}^T x = y^* x$.

A vector space doesn't need an inner product, but if it has one, it is an inner product space, and it automatically gets some geometry: an inner product defines a length (*the* natural norm for the inner product space) as

$$\|x\| \equiv \sqrt{<x, x>},$$

and the angle, α, between vectors by

$$<x, y> = \|x\| \|y\| \cos(\alpha).$$

You might remember this formula as a theorem from Euclidean geometry; the difference is that here we are *defining* angles through this formula. Among some immediate consequences are the Cauchy-Schwartz inequality:

$$<x, y><y, x> \quad \leq \quad <x, x><y, y>$$

the parallelogram identity:

$$\|x + y\|^2 + \|x - y\|^2 \; = \; 2(\|x\|^2 + \|y\|^2),$$

and a geometric interpretation of nullspace, called orthogonality:

$$x \perp y \quad \Leftrightarrow \quad <x, y> = 0.$$