Example: Sensitivity of Linear Systems

Consider the vectors
$$a_1 = \begin{pmatrix} 1000 \\ 999 \end{pmatrix}, \text{ and } a_2 = \begin{pmatrix} 1001 \\ 1000 \end{pmatrix}.$$
Let's let the matrix A have columns a_1 and a_2:
$$A = \left[\begin{array}{cc} a_1, & a_2 \end{array} \right] = \left[\begin{array}{cc} 1000 & 1001 \\ 999 & 1000 \end{array} \right].$$

You can check that A is nonsingular. Then a_1 and a_2 are linearly independent, and so any $b \in \mathbb{R}^2$ can be written in exactly one way as a linear combination $b = x_1 a_1 + x_2 a_2$. The coefficients x_1 and x_2 of this combination are the coordinates of the solution $x = (x_1, x_2)^T$, of the matrix equation $Ax = b$.

Now let's take b to be
$$b = \begin{pmatrix} 2001 \\ 1998.9 \end{pmatrix}.$$
Solving $Ax = b$, we find that the (exact) unique solution is
$$x = \begin{pmatrix} 101.1 \\ -99 \end{pmatrix}.$$
that is,
$$b = 101.1\ a_1 - 99\ a_2,$$

Now suppose we round b by about 1 part in 20,000:
$$\tilde{b} = \begin{pmatrix} 2001 \\ 1999 \end{pmatrix}.$$

This small change in b can be measured in vector norms: $\|b - \tilde{b}\| / \|b\| \approx 0.00005$. Now how much of a_1 and a_2 do we need to make $\tilde{b}$? Well, we solve $Ax = \tilde{b}$ to get
$$\tilde{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$
that is,
$$\tilde{b} = 1\ a_1 + 1\ a_2.$$
A (relative) change in b of about 10^{-4} gives a change in x of over 10^2.

This example was designed to make a point: A is very close to a singular matrix (although $\det(A)$ doesn't tell us that). Singular systems either have no solution or have infinitely many solutions, depending on the rhs.

We will find find ways to measure how close a matrix is to the nearest singular matrix – and when it is close to being singular we should view any 'solution' to $Ax = b$ skeptically...