Example: Sensitivity of Linear Systems

Consider the vectors

\[ a_1 = \begin{pmatrix} 0.999 \\ 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 \\ 1.001 \end{pmatrix}. \]

Let’s let the matrix \( A \) have columns \( a_1 \) and \( a_2 \):

\[ A = [ a_1, \ a_2 ] = \begin{bmatrix} 0.999 & 1 \\ 1 & 1.001 \end{bmatrix}. \]

You can check that \( A \) is nonsingular. Then \( a_1 \) and \( a_2 \) are linearly independent, and so any \( b \in \mathbb{R}^2 \) can be written in exactly one way as a linear combination \( b = x_1 a_1 + x_2 a_2 \). The coefficients \( x_1 \) and \( x_2 \) of this combination are the coordinates of the solution \( x = (x_1, x_2)^t \), of the matrix equation \( Ax = b \).

Now let’s take \( b \) to be

\[ b = \begin{pmatrix} 1.9989 \\ 2.0010 \end{pmatrix}. \]

When we solve \( Ax = b \), we find that (no rounding errors here)

\[ x = \begin{pmatrix} 101.1 \\ -99 \end{pmatrix}. \]

that is,

\[ b = 101.1 \ a_1 + 99 \ a_2, \]

Now suppose we round \( b \) to the nearest thousandths place:

\[ \tilde{b} = \begin{pmatrix} 1.999 \\ 2.001 \end{pmatrix}. \]

This small change in \( b \) can be measured: \( \|b - \tilde{b}\|_\infty = 0.0001 \). Now how much of \( a_1 \) and \( a_2 \) do we need to make \( \tilde{b} \)? Well, we solve \( Ax = \tilde{b} \) to get

\[ x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \]

that is,

\[ \tilde{b} = 1 \ a_1 + 1 \ a_2. \]

A change in \( b \) of about \( 10^{-4} \) gives a change in \( x \) of over \( 10^2 \).

This example was designed to make a point, you can see how it works by interpreting \( Ax = b \) as the intersection of 2 lines. Go ahead and plot the lines...