

The first (and only?) step of Gaussian elimination

Let $A^{(1)} = M_1 A$ be the matrix obtained by using the first row of A to eliminate (send to zero) the entries below a_{11} . Recall that M_1 is of the form $M_1 = I + m_1 e_1^T$, and that we want

$$M_1 A e_1 = a_{11} e_1.$$

Then $a_{11} e_1 = (I + m_1 e_1^T) A e_1 = A e_1 + m_1 e_1^T A e_1 = A e_1 + a_{11} m_1$, and if $a_{11} \neq 0$, we can solve this for m_1 :

$$m_1 = (\frac{1}{a_{11}})(a_{11} e_1 - A e_1).$$

This is simply a vector notation for $m_{11} = 0$ and $m_{i,1} = -a_{i,1}/a_{11}$, $i = 2, 3, \dots, n$.

Once we know m_1 , we can complete step 1 of GE, by computing $A^{(1)} = M_1 A$:

$$A^{(1)} = A + m_1 (e_1^T A).$$

We do *not* explicitly form M_1 (why?), we *do* respect those parentheses (why?), and we do *not* explicitly multiply e_1^T and A (why?). If one pays attention to these admonitions, then m_1 and $A^{(1)}$ can be computed in 1 division, $n - 1$ multiplications, $(n - 1)^2$ multiplications, and $(n - 1)^2$ additions, for a total of $2n^2 + O(n)$ flops.

We are finished with step 1 of GE, and partition $A^{(1)}$ as

$$A^{(1)} = \begin{bmatrix} a_{11} & \tilde{a}_1^T \\ 0 & \tilde{A}^{(1)} \end{bmatrix}.$$

Please stop reading for a bit and think about step 2... Isn't step 2 going to be just like step 1, but with the names changed? Won't we use the first row of $\tilde{A}^{(1)}$ to eliminate the entries below $\tilde{a}_{11}^{(1)}$? Then you know how it goes. In fact, you know how it goes for step $k + 1$: Suppose we've completed k steps ($\tilde{A}^{(k)}$ is $(n - k) \times (n - k)$):

$$A^{(k)} = \begin{bmatrix} U^{(k)} & X^{(k)} \\ 0 & \tilde{A}^{(k)} \end{bmatrix}.$$

You know the formulas,

$$\tilde{m}_{k+1} = (\frac{1}{\tilde{a}_{11}})(\tilde{a}_{11} e_1 - \tilde{A}^{(k)} e_1) \quad \text{and} \quad \hat{A}^{(k+1)} = \tilde{A}^{(k)} + \tilde{m}_{k+1} (e_1^T \tilde{A}^{(k)}).$$

We get the $\tilde{A}^{(k+1)}$ simply by partitioning $\hat{A}^{(k+1)}$ as we partitioned $A^{(1)}$ above.

Computing $\tilde{A}^{(k+1)}$ requires $2(n - k)^2 + O(n - k)$ flops, so the flop count for all $n - 1$ steps is

$$\sum_{k=1}^{n-1} (2(n - k)^2 + O(n - k)) = \frac{2}{3} n^3 + O(n^2).$$

The relationship of M_k to \tilde{M}_k is $m_k^T = [0 \quad \tilde{m}_k^T]$ and

$$I + m_k e_k^T = M_k = \begin{bmatrix} I & 0 \\ 0 & \tilde{M}_k \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I + \tilde{m}_k e_1^T \end{bmatrix}.$$