The Action of Householder Reflectors

Let $A \in \mathbb{R}^{m \times n}$, $m \geq n$. You may recall that the Householder QR factorization can be written as

$$H_p \cdots H_2 H_1 A = R,$$

where $p = \min(n, m-1)$ and $H_k = I - (\frac{2}{u_k^T u_k}) u_k u_k^T$ is a Householder reflector which introduces zeros into positions $k+1$ to m of the k^{th} column of the matrix $A^{(k-1)} = H_{k-1} \cdots H_2 H_1 A$. This gives $A = QR$, where we define

$$Q = H_1 H_2 \cdots H_p.$$

The H_j are not explicitly formed, since that is very inefficient, and since they are completely determined by the u vectors. The matrix Q is rarely formed, either. What we usually do is save the u_j's. If we let Q_u be the array whose j^{th} column is u_j, it would be called the "factored form" of Q. In Matlab, if u is our variable name for u_j, then we might write $Qu(:, j) = u$. In a memory efficient code, we would only save entries $j : m$ of u_j ($\tilde{u}_j = u_j(j : m)$), and would use the lower triangle of A (plus an n-vector) to store $\tilde{u}_j,\ j = 1, \ldots, p$.

If later we will need to find the vector $c = Q^T b$, then instead of explicitly forming Q, we use the routine HOUSEQACT: Notice that

$$C = Q^T B = H_p \cdots H_2 H_1 B,$$

so we write HOUSEQACT to form this product

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C = B;
for j=1:p,
    C = Hj * C; %but you code this line the right way...
end
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Of course, it would be just as easy to write code to implement

$$C = QB = H_1 H_2 \cdots H_p B,$$

and BQ^T or BQ. In fact, while it is rarely needed explicitly, we could compute Q above as $Q = QI$, or $Q = IQ$,...

We discuss how to find the u_j's and how to code the algorithm above efficiently for speed and memory on other pages, but a good implementation of the loop above should be all the convincing you need to understand that Householder reflectors are never explicitly computed. If $C \in \mathbb{R}^{m \times r}$, a good implementation of the pseudocode above requires $r \sum_{j=0}^{n} 4(m-j) \approx 2mnr(2 - \frac{n}{m})$ flops, and even if we knew $Q \in \mathbb{R}^{m \times m}$ explicitly, that matrix product would require $2m^2 r$ flops (which is more costly than HOUSEQACT since $n \leq m$).