Coordinates

Let R be a linear space with basis $\mathcal{B} = \{x_1, x_2, \ldots, x_n\}$. Then if x is any vector in R, we can write $x = \sum_{i=1}^{n} c_i x_i$, and the scalars c_i are unique. Thus, for this basis, each $x \in R$ has associated with it a unique n-tuple $(c_1, c_2, \ldots, c_n)$. This n-tuple is called *the coordinate vector for x wrt $\mathcal{B}$*, and the c_i are *the coordinates of x wrt $\mathcal{B}$*. A suggestive notation is $[x]_{\mathcal{B}} = (c_1, c_2, \ldots, c_n)^T$.

So x is the (possibly abstract) vector, and $[x]_{\mathcal{B}}$ is the coordinate vector for x; not abstract at all, but a sequence of n scalars. If $\mathcal{B} = \{e_1, e_2, \ldots, e_n\}$, the standard ordered basis for $\mathbb{R}^n$, then the vector $(2, -1, 3)^T = 2e_1 - 1e_2 + 3e_3$, and its coordinates are $(c_1, c_2, c_3) = (2, -1, 3)$ (after all, this is the *standard* ordered basis for $\mathbb{R}^3$). But now, if we take as our basis $\mathcal{B} = \{e_1, e_3, e_2\}$, then the coordinate vector of $x = (2, -1, 3)^T$ is $[x]_{\mathcal{B}} = (2, 3, -1)^T$.

So how do we find the coordinates for a vector x with respect to some basis $\mathcal{B}$? Well, if $\mathcal{B} = \{x_1, x_2, \ldots, x_n\}$, then we want to find the c_i in $x = \sum_{i=1}^{n} c_i x_i$. If we are in $\mathbb{R}^n$, then we can simply construct the matrix $P = [x_1, x_2, \ldots, x_n]$, whose jth column is x_j. Then we have the nonsingular system of linear equations $P[x]_{\mathcal{B}} = x$, whose solution is $[x]_{\mathcal{B}} = P^{-1}x$. If now we want to change basis, that is, represent x in the basis $\mathcal{B}' = \{y_1, y_2, \ldots, y_n\}$, then we construct $S = [y_1, y_2, \ldots, y_n]$ and solve $P[x]_{\mathcal{B}} = x = S[x]_{\mathcal{B}'}$ to get $[x]_{\mathcal{B}'} = S^{-1}P[x]_{\mathcal{B}}$. Notice that the jth column of $S^{-1}P$ is $[x_j]_{\mathcal{B}'}$. $S^{-1}P$ is called the change of basis matrix from $\mathcal{B}$ to $\mathcal{B}'$.

Let's move on to linear operators and matrices. In a finite dimensional linear space we can also give linear operators coordinates. The coordinate representation of a linear operator is a matrix. Let U and V be linear spaces with bases $\mathcal{U} = \{u_1, u_2, \ldots, u_n\}$ and $\mathcal{V} = \{v_1, v_2, \ldots, v_m\}$, respectively. Suppose $L : U \longrightarrow V$ is linear. We would like to represent L as the matrix $[L]_{\mathcal{V},\mathcal{U}}$ in such a way that if $v = L(u)$, then $[v]_{\mathcal{V}} = [L]_{\mathcal{V},\mathcal{U}}[u]_{\mathcal{U}}$. If we let $[v]_{\mathcal{V}} = [c_i]$, $[u]_{\mathcal{U}} = [b_j]$, and $[L]_{\mathcal{V},\mathcal{U}} = [a_{ij}]$, then we must have $c_i = \sum_{j=1}^{n} a_{ij} b_j, \quad i = 1, 2, \ldots, m$. In particular, this must hold for $u = u_j$ (where all $b's$ are zero but the jth): $c_i = a_{ij} b_j, \quad i = 1, 2, \ldots, m$. So the jth column of $[L]_{\mathcal{V},\mathcal{U}}$ must be the coordinates of $L(u_j)$ in the basis $\mathcal{V}$, i.e. $[L]_{\mathcal{V},\mathcal{U}} e_j = [L(u_j)]_{\mathcal{V}}$; and this completely describes $[L]_{\mathcal{V},\mathcal{U}}$.

Now consider the identity map from U to U, given by $\mathcal{I}(u) = u$. The matrix for I in the basis $\mathcal{U}$ has as its jth column $[u_j]_{\mathcal{U}}$, so $[\mathcal{I}]_{\mathcal{U},\mathcal{U}} = I$, the identity matrix. What about $[\mathcal{I}]_{\mathcal{V},\mathcal{U}}$? Well, the jth column of $[\mathcal{I}]_{\mathcal{V},\mathcal{U}}$ is $[u_j]_{\mathcal{V}}$, which is precisely the change of basis matrix $S^{-1}P$ that we found above! Also, $[\mathcal{I}]_{\mathcal{V},\mathcal{U}}[I]_{\mathcal{U},\mathcal{V}} = [\mathcal{I}]_{\mathcal{V},\mathcal{V}} = I$, so the change of basis matrix from $\mathcal{U}$ to $\mathcal{V}$ is the inverse of the change of basis matrix from $\mathcal{V}$ to $\mathcal{U}$, and similarity is simply $[L]_{\mathcal{U},\mathcal{U}} = [\mathcal{I}]_{\mathcal{U},\mathcal{V}}[L]_{\mathcal{V},\mathcal{V}}[\mathcal{I}]_{\mathcal{V},\mathcal{U}}$.