Concrete Eigenstuff

Square matrices in $\mathbb{R}^{n \times n}$ are linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$. The standard ordered basis for $\mathbb{R}^n$ consists of the columns of I: $\mathscr{SOB} = \{e_1, e_2, \ldots, e_n\}$. When we are looking at a vector v as an n-tuple, we are, by default, seeing it as coordinates in $\mathscr{SOB}$, i.e. $v = [v]_{\mathscr{SOB}}$. If $\mathscr{B} = \{v_1, v_2, \ldots, v_n\}$ is another basis for $\mathbb{R}^n$, v can be written (uniquely) as some linear combination of the v_k's:

$$v = \sum_{i=1}^{n} c_i v_i.$$

The coordinate vector of v wrt $\mathscr{B}$ is $[v]_{\mathscr{B}} = (c_1, c_2, \ldots, c_n)^T \in \mathbb{R}^n$.

Now we can change from the standard basis to $\mathscr{B}$ by finding the coordinate vectors of e_k wrt $\mathscr{B}$ (solving a linear system):

$$[e_k]_{\mathscr{B}} = (c_{1k}, c_{2k}, \ldots, c_{nk})^T.$$

The matrix $C = [c_{ij}]$ is such that $[v]_{\mathscr{B}} = Cv$; it may even be suggestive to write $C = [I]_{\mathscr{B},\mathscr{SOB}}$, so that $[v]_{\mathscr{B}} = Cv = [I]_{\mathscr{B},\mathscr{SOB}}[v]_{\mathscr{SOB}}$. Changing from $\mathscr{B}$ to $\mathscr{SOB}$ is always trivial (solving a linear system with coefficient matrix I):

$$[v_k]_{\mathscr{SOB}} = (v_{1k}, v_{2k}, \ldots, v_{nk})^T.$$

And if V is the matrix with j^{th} column v_j, we have

$$[I]_{\mathscr{SOB},\mathscr{B}} = C^{-1} = V \qquad \& \qquad [I]_{\mathscr{B},\mathscr{SOB}} = C = V^{-1},$$

Now if $A = [A]_{\mathscr{SOB},\mathscr{SOB}}$ is an $n \times n$ matrix, then what is $B = [A]_{\mathscr{B},\mathscr{B}}$, the coordinate representation of A in the new basis $\mathscr{B}$? The notation developed above makes the question easy to answer: $A = IAI$, so

$$B \quad = \quad [A]_{\mathscr{B},\mathscr{B}} \quad = \quad [I]_{\mathscr{B},\mathscr{SOB}}[A]_{\mathscr{SOB},\mathscr{SOB}}[I]_{\mathscr{SOB},\mathscr{B}} \quad = \quad V^{-1}AV.$$

However you first met similarity, this is her true face.

Quite often it is very helpful to change the variables in a problem to some more natural set of variables (you have seen the efficacy of polar and spherical coordinates, moving coordinate systems, Fourier transforms, etc.). In fact it is routine, and for square matrices (linear operators) it is similarity. If A and B are similar, then they represent the same function, but wrt different bases.

A *is* what A *does*. How is Ax related to x? A stretches some directions and attenuates others. Is there a basis for $\mathbb{R}^n$ wrt which this is easy to see? About the simplest type of matrix we have is a diagonal matrix. If B is diagonal, then $Be_k = b_{kk}e_k$, and $B = V^{-1}AV$ from above requires $AV = VB$. The k^{th} column of $AV = VB$ is

$$Av_k = b_{kk}v_k$$