The residual vector for $Ax = b$

Suppose $A \in \mathbb{R}^{n \times n}$ is nonsingular, so that $x = A^{-1}b$ is the unique solution to $Ax = b$ and x solves $Ax = b$ if and only if the residual vector, $r = b - Ax$, satisfies $r = 0$. Let $\bar{x}$ be a computed approximation to x, and define

$$\bar{r} = b - A\bar{x}.$$

A measure (in units of $\|b\|$) of how much $\bar{x}$ fails to satisfy $Ax = b$ is simply

$$\rho(\bar{x}) = \frac{\|\bar{r}\|}{\|b\|}. \tag{1}$$

This number, sometimes called the *relative residual*, might be the quantity you are interested in, but often we care about how well $\bar{x}$ approximates the true solution x. It is important to note here that $\bar{r}$ and ρ are quantities that we can compute from A, b, and $\bar{x}$, but x is forever unknown. We investigate below rhe relationship between $\rho(\bar{x})$ and both the actual error and the *backward error*.

We only require that the norm(s) being used are consistent, i.e. $\|Ax\| \leq \|A\|\|x\|$ for any $x \in \mathbb{R}^n$. We call $\kappa(A) \equiv \|A\|\|A^{-1}\|$ the condition number of A (more specifically, it is a relative condition number for the problem "given A, find A^{-1}").

Notice that $A^{-1}\bar{r} = A^{-1}b - A^{-1}A\bar{x}$, so

$$x - \bar{x} = A^{-1}\bar{r}.$$

From this we easily get the (relative) error bound

$$\epsilon(\bar{x}) \equiv \frac{\|x - \bar{x}\|}{\|x\|} = \frac{\|A^{-1}\bar{r}\|}{\|x\|} \leq \frac{\|A^{-1}\|\|\bar{r}\|}{\|x\|} \leq \frac{\|A^{-1}\|\|A\|\|\bar{r}\|}{\|b\|} = \kappa(A)\frac{\|\bar{r}\|}{\|b\|}.$$

From the other side:

$$\frac{1}{\kappa(A)}\frac{\|\bar{r}\|}{\|b\|} = \frac{\|b - A\bar{x}\|}{\|A\|\|A^{-1}\|\|b\|} \leq \frac{\|A(A^{-1}b - \bar{x})\|}{\|A\|\|x\|} \leq \frac{\|x - \bar{x})\|}{\|x\|} = \epsilon(\bar{x}),$$

together giving the tidy result

$$\frac{\rho(\bar{x})}{\kappa(A)} \leq \epsilon(\bar{x}) \leq \kappa(A)\rho(\bar{x}). \tag{2}$$

Finally we discuss the practical idea of backward error: Does $\bar{x}$ solve a system close to $Ax = b$? More specifically, what's the smallest change we need to make to $Ax = b$ so that $\bar{x}$ is a solution? Define the (relative) backward error in $\bar{x}$ as

$$\beta(\bar{x}) \equiv \min_{\delta A, \delta b} \{\|\delta A\|/\|A\| + \|\delta b\|/\|b\|\}, \quad \text{subject to } (A + \delta A)\bar{x} = b + \delta b.$$

Now notice that $\bar{x}$ exactly satisfies the linear system $Ax = b - \bar{r}$ (this is just the definition of $\bar{r}$). So taking $\delta A = 0$ and $\delta b = -r$, we know that $\beta(\bar{x}) \leq \|\bar{r}\|/\|b\|$ (this is because $(\delta A, \delta b) = (0, -r)$ is a feasible point for the minimization). Therefore,

$$\beta(\bar{x}) \leq \rho(\bar{x}). \tag{3}$$

Take these numbered equations with you...