Arnoldi

Given
$$A \in \mathbb{R}^{n\times n}, \quad b \in \mathbb{R}, \quad \|b\|_2 = \alpha_0$$
Let's investigate the Gram-Schmidt orthogonalization of the columns of
$$C = [b,\ Ab,\ \ \ldots,\ \ A^{m-1}b].$$
The Arnoldi method does this, giving a Hessenberg matrix H such that
$$AV = VH + we_m^T, \quad V^TV = I_m, \quad V^Tw = 0.$$
Let's assume that $h_{j+1,j} \neq 0,\ j = 1, 2, \ldots, m-1$ (thus H is *unreduced*). The orthogonal projector onto the column space of C is $V(V^TV)^{-1}V^T = VV^T$, thus
$$VV^TA^kb = A^kb, \quad k \leq m-1,$$
and the Gram-Schmidt QR factorization we wanted is
$$C = VR, \quad \text{with} \quad R \equiv \alpha_0[e_1,\ He_1,\ \ \ldots,\ \ H^{m-1}e_1] \ \text{ and } \ r_{ii} = \alpha_0 \prod_{j=1}^{m-1} h_{j+1,j}.$$
Now
$$HRe_k = Re_{k+1}, \quad k = 1, 2, \ldots m-1,$$
means the companion matrix (rational canonical form) for H is
$$F = R^{-1}HR.$$
So, if $q_{_H}$ is the characteristic polynomial of H, then
$$q_{_H}(x) = x^m - \sum_{i=0}^{m-1} c_i x^i, \quad \text{where } c = R^{-1}HRe_m.$$
Finally, if $p(x) = x^m - \sum_{i=0}^{m-1} a_i x^i$, then $p(A)b = ACe_m - Ca$, and remarkably
$$q_{_H} = \operatorname*{argmin}_{p\in\mathcal{M}onic_m} \|p(A)b\|_2.$$

We have just connected a Hessenberg form, a QR factorization, a companion form, and a variational principle all under the Arnoldi umbrella, but can't resist squeezing in a few more players: An unreduced Hessenberg matrix with multiple eigenvalues is not diagonalizable. Thus, if our A is diagonalizable, then its eigenvalues are distinct. Suppose $AX = X\Lambda$ is a diagonalization of A. Then
$$\begin{aligned} C &= [b,\ (X\Lambda X^{-1})b\ , (X\Lambda X^{-1})^2b, \ldots,\ (X\Lambda X^{-1})^{m-1}b] \\ &= X[X^{-1}b,\ \Lambda X^{-1}b,\ \Lambda^2X^{-1}b, \ldots,\ \Lambda^{m-1}X^{-1}b] \\ &= X\ \mathrm{diag}(X^{-1}b)\ [e,\ \Lambda e,\ \Lambda^2e, \ldots,\ \Lambda^{m-1}e] \\ &\equiv YW. \end{aligned}$$
Hopefully you recognize the rightmost factor W as a Vandermonde matrix on the eigenvalues $\lambda_1,\ \lambda_2, \ldots,\ \lambda_n$. Now if we take $m = n$, $F = R^{-1}HR = C^{-1}AC$ is a companion form of A, whose eigenvalues are Λ, while the eigenvectors of A are the columns of $Y = X\ \mathrm{diag}(X^{-1}b) = CW^{-1}$.