Name: ____________________________

(15) 1. Let $A \in \mathbb{R}^{n\times n}$ be full rank (non-singular) and $b \in \mathbb{R}^n$. Let $\bar{x}$ be a computed approximate solution to the problem "Solve $Ax = b$", and define $r = b - A\bar{x}$ (the *residual*).

(a) What can you say about the accuracy of $\bar{x}$, if $\frac{\|r\|}{\|b\|}$ is small? (giving an inequality is ok here)

(b) What, if anything, does $\frac{\|r\|}{\|b\|}$ tell us about the backward stability of $\bar{x}$?

(c) What, if anything, does $\frac{\|r\|}{\|b\|}$ tell us about the conditioning of "Solve $Ax = b$"?

(10) 2. Let $A \in \mathbb{R}^{m\times n}$ with $m > n$. Describe the singular value decomposition of A, giving as much detail as you can. (not the process, but the properties of the matrices in the factorization).

(20) 3. Let $A \in \mathbb{R}^{m \times n}, \quad m > n$ and let $b \in \mathbb{R}^m$. Let the columns of A be linearly independent. Consider the least squares problem

$$\arg\min_x \|Ax - b\|_2 \qquad \text{(LS)}.$$

(a) Describe the explicit normal equations approach to solving (LS). Include the cost in flops for each step.

(b) Explain any advantages or disadvantages of the normal equations approach compared to QR methods like MGS or HQR.

(c) What is the condition number of "solve $A^T Ax = A^T b$"?

(5) 4. Let $A \in \mathbb{R}^{n \times n}$ and $u, x \in \mathbb{R}^{n \times 1}$. Give an efficient algorithm for computing $B = ux^T Axu^t$ (this can be pseudo-code or simply writing B in a form for which parantheses indicates the algorithm). Approximately how many flops does your method require?

(25) 5. Let $A \in \mathbb{R}^{m\times n}, \quad m > n$ be full rank.

(a) Describe the Gram-Schmidt QR factorization of A (not the process, but the properties of Q and R).

(b) Explain how to solve (LS) using the Gram-Schmidt QR factorization of A.

(c) Describe the Householder QR factorization of A (not the process, but the properties of Q and R).

(d) Explain how to solve (LS) using the Householder QR factorization of A.

(20) 6. Conditioning

(a) Explain what we mean by a well-conditioned problem without referring to condition number.

(b) What is the condition number for "solve $Ax = b$"? Is this a relative or absolute condition number?

(c) What is the condition number for "solve $\arg\min \|b - Ax\|_2$? Is this a relative or absolute condition number?

(d) How is $\kappa_2(A)$ related to the singular values of A? (give a formula)

(5) 7. If you know that the initial data for a computational problem has relative accuracy of 10^{-16}, and the problem has a relative condition number $\kappa = 10^9$, then how many correct (decimal) digits could you hope to have in your computed solution?