Name: ______________________________

(12) 1. Let $A \in \mathbb{R}^{n \times n}$ be full rank (non-singular) and $b \in \mathbb{R}^n$. Let $\bar{x}$ be an approximate solution to the problem "Solve $Ax = b$", and define $r = b - A\bar{x}$ (the *residual*).

(a) Give a condition number for "Solve $Ax = b$". Is this a relative or absolute condition number?

(b) Derive a bound for the relative error in $\bar{x}$ in terms of $\||b\|$, $\|r\|$, and $\kappa(A)$.

(9) 2. Let $A = \begin{bmatrix} -1 & 0 & 3 & -5 \\ 2 & 0 & 6 & -6 \end{bmatrix}$.

(a) Compute $\|A\|_1$.

(b) Compute $\|A\|_\infty$.

(c) Compute $\|A\|_F$.

(22) 3. Let $A \in \mathbb{R}^{m \times n}, \quad m > n$ be full rank.

(a) Describe the Gram-Schmidt (GS) QR factorization of A (not the process or storage scheme, but sizes and properties of the exact Q_{GS} and R_{GS} matrices).

(b) Describe the Householder (HQR) QR factorization of A (not the process or storage scheme, but sizes and properties of the exact Q_{HQR} and R_{HQR} matrices).

(c) What do we mean by the "factored form" of Q_{HQR}?

(d) Discuss any relationships between Q_{GS} and Q_{HQR}.

(9) 4. Let $x = (-1, 0, 3, -4)^T$.

(a) Compute $\|x\|_1$.

(b) Compute $\|x\|_2$.

(c) Compute $\|x\|_\infty$.

(16) 5. General conditioning and stability

(a) Explain what we mean by an ill-conditioned problem without referring to condition number.

(b) Explain how the ideas of conditioning and backward error can give us an estimate of the error in a computational problem.

(c) If you know that the initial data for a computational problem has relative accuracy of 10^{-8}, and the problem has relative condition number $\kappa = 10^3$, then what is the best possible relative error you would hope to achieve in your computed solution?

(8) 6. Let $A, B \in \mathbb{R}^{n \times n}$ and $u, x \in \mathbb{R}^{n \times 1}$. Give an efficient algorithm for computing $C = B + ux^T Axu^T$ (this can be pseudo-code or simply writing B in a form for which parantheses indicates the algorithm). Approximately how many flops does your method require?

(24) 7. Let $A \in \mathbb{R}^{m \times n}$, $m > n$, be full rank, and let $b \in \mathbb{R}^m$. Consider the least squares problem

$$\min_x \|Ax - b\|_2 \qquad \text{(LS)}.$$

(a) Describe coarsely (5 or fewer steps) the explitic normal equations approach to solving (LS). Include the cost in flops for each step.

(b) Describe coarsely (5 or fewer steps) the Gram-Schmidt QR approach to solving (LS). Include the cost in flops for each step.

(c) Describe coarsely (5 or fewer steps) the Householder QR approach to solving (LS). Include the cost in flops for each step.

(d) Which, if any, of the above methods is backward stable for (LS)?