Convergence of the secant method

The secant iteration uses a secant line approximation to the function f to approximate its root. Let p be such that $f(p) = 0$, and let p_{k-1} and p_k be two approximations to p. Let us use the abbreviation $f_k \equiv f(p_k)$ throughout. If we take as our next approximation to p the root of the (secant) line passing through (p_{k-1}, f_{k-1}) and (p_k, f_k), then we get the secant method:

$$p_{k+1} = p_k - \frac{p_k - p_{k-1}}{f_k - f_{k-1}}\, f_k$$

(You can also arrive at this method by starting with Newton's method and then replacing $f'(p_k)$ by $\frac{f_k - f_{k-1}}{p_k - p_{k-1}}$.)

We are interested here in analyzing the speed of convergence of the secant method. To that end, define the error at the k^{th} step to be $e_k = p_k - p$. We will use a Taylor approximation $f(p + e_k) = e_k f'(p) + e_k^2 f''(p)/2 + O(e_k^3)$ under the assumption that $f'(p) \neq 0$. Then

$$\begin{aligned}
e_{k+1} &= p_{k+1} - p \;=\; p_k - \frac{p_k - p_{k-1}}{f_k - f_{k-1}}\, f_k - p \\
&= \frac{(p_{k-1} - p) f_k - (p_k - p) f_{k-1}}{f_k - f_{k-1}} \\
&= \frac{e_{k-1} f_k - e_k f_{k-1}}{f_k - f_{k-1}} \\
&= \frac{e_{k-1} f(p + e_k) - e_k f(p + e_{k-1})}{f(p + e_k) - f(p + e_{k-1})} \\
&= \frac{e_{k-1}(e_k f'(p) + e_k^2 f''(p)/2 + O(e_k^3)) - e_k(e_{k-1} f'(p) + e_{k-1}^2 f''(p)/2 + O(e_{k-1}^3))}{e_k f'(p) + e_k^2 f''(p)/2 + O(e_k^3) - (e_{k-1} f'(p) + e_{k-1}^2 f''(p)/2 + O(e_{k-1}^3))} \\
&= \frac{e_{k-1} e_k f''(p)(e_k - e_{k-1})/2 + O(e_{k-1}^4)}{(e_k - e_{k-1})(f'(p) + (e_k + e_{k-1}) f''(p)/2 + O(e_{k-1}^2))} \\
&= \frac{e_{k-1} e_k f''(p)}{2 f'(p)} + O(e_{k-1}^3)
\end{aligned}$$

We want to find α such that $|e_{k+1}| \to C|e_k|^\alpha$, so we drop the cubic term and solve

$$\left|\frac{e_{k-1} e_k f''(p)}{2 f'(p)}\right| = C|e_k|^\alpha,$$

giving $|e_{k+1}|^{\alpha-1} = D|e_k|$, where $D = |f''(p)/(2Cf'(p))|$. Then $|e_{k+1}|^{\alpha(\alpha-1)} = D^\alpha |e_k|^\alpha$. This forces $C = D^\alpha$ and $\alpha(\alpha - 1) = 1$. The only positive solution to the equation in α is $\alpha = (1 + \sqrt{5})/2 \approx 1.618$ (the golden mean). Furthermore, the asymptotic error constant must be $C = |f''(p)/(2f'(p))|^{\alpha-1} \approx |\frac{f''(p)}{2f'(p)}|^{.618}$.