Conditioning and Stability

A problem is *well conditioned* if a small change in the input creates a small change in the output (solution).

A computation is *backward stable* if it produces the exact solution to a nearby problem.

There is room for debate in this definition, but it does succinctly capture the idea. (I like better: A computation is *backward stable* if the result is *close* to the exact solution to a nearby problem). In any case, we say that a method is backward stable for a set of problems if it is backward stable for each problem in the set.

If the problem we are trying to solve has a unique solution, then we can usually formulate it as "evaluate $f(x)$", where x represents the input and $f(x)$ represents the solution (output). Let's represent our input space by $\mathcal{D}$ and our output space by $\mathcal{R}$. Then our computed result can be represented by $\bar{f} : \mathcal{D} \longrightarrow \mathcal{R}$, where the exact result is represented by $f : \mathcal{D} \longrightarrow \mathcal{R}$. That is, $\bar{f}(x)$ is our computed approximation to $f(x)$.

A problem is *well conditioned* in $\mathcal{D}$ if for all small δ with $x + \delta \in \mathcal{D}$, $f(x + \delta)$ is close to $f(x)$.

A method is *backward stable* if for all $\bar{f}(x) \in \mathcal{R}$, there exists a small ϵ with $x + \epsilon \in \mathcal{D}$ and such that $\bar{f}(x)$ is close to $f(x + \epsilon)$.

Note that conditioning has nothing to do with methods and that stability has nothing to do with problems. This decomposition of our computation into the independent ideas of *stability-of-the-method* and *conditioning-of-the-problem* is fundamental to modern scientific computation.

Now suppose we use a backward stable method to solve a well conditioned problem. Then since the method is backward stable there is a small ϵ such that $\bar{f}(x)$ is close to $f(x + \epsilon)$, and since the problem is well conditioned $f(x + \epsilon)$ is close to $f(x)$. Thus, $\bar{f}(x)$ (the computed solution) is close to $f(x)$ (the exact solution).

Plainly spoken: a backward stable method applied to a well conditioned problem gives an accurate solution.

Here we have used the words *small* and *close* to give us the freedom to consider either absolute or relative errors and to blurr the lines between good/bad and easy/hard according to our application.

As an example, consider the addition of 2 real numbers. Here we will say $f(x, y) = x + y$ using the method $\bar{f}(x, y) = \mathrm{fl}(\mathrm{fl}(x) + \mathrm{fl}(y))$. A backward rounding error analysis shows that

$$\begin{aligned} \bar{f}(x, y) &= (x(1 + \delta_x) + y(1 + \delta_y))(1 + \delta_+) \\ &= x(1 + \epsilon_x) + y(1 + \epsilon_y) \\ &= f(x(1 + \epsilon_x), y(1 + \epsilon_y)), \text{ where } |\epsilon_x|, |\epsilon_y| \leq 2\mu + O(\mu^2). \end{aligned}$$

Thus, the addition of two real numbers is backward stable (in a relative sense, at least).

On the other hand, if $|x + y|$ is small, then we know that small changes in x and/or y can lead to big relative changes in $x + y$. This means the addition of two real numbers *can* be illconditioned (this is digit cancellation from a conditioning perspective!).