Condition Numbers

A problem is *well conditioned* if a small change in the input (the data) *always* creates a small change in the output (the solution).

A problem is *ill-conditioned* if a small change in the input (data) *can* create a large change in the output (solution).

There is actually a continuum here, ranging from the extremely well conditioned (e.g. "evaluate the fcn $f(x) \equiv 1$") to the extremely ill-conditioned (e.g. "evaluate a fcn at a discontinuity"). In order to quantify the notion, we will define a *condition number*.

A condition number is simply a number which describes how ill- or well conditioned a problem is; the bigger the number the more ill-conditioned the problem. Ideally, the condition number will behave as follows:

$$\text{size of change in solution} \quad = \quad \text{condition number} \quad \cdot \quad \text{size of change in input}$$

If we have norms, we can define an absolute condition number, ν, which satisfies

$$\|\text{ change in solution }\| \quad = \quad \nu \ \cdot \ \|\text{ change in input }\|,$$

or a relative condition number, κ, which satisfies

$$\frac{\|\text{ change in solution }\|}{\|\text{ solution }\|} \quad = \quad \kappa \ \cdot \ \frac{\|\text{ change in input }\|}{\|\text{ input }\|}.$$

In most cases, quantities like ν or κ above are impossible (or silly) to compute; impossible even to define, except in the abstract terms above, for these true condition numbers are functions like $\nu = \nu($ input, change in input, solution, change in solution $)$, and we just don't know these quantities.

But that's ok. Because usually what is desired is a (hopefully easily computed) approximation to the condition number, which will satisfy

$$\|\text{ change in solution }\| \quad \approx \quad \bar{\nu} \ \cdot \ \|\text{ change in input }\|,$$

(in the case of an absolute condition estimator). To get this kind of result we usually restrict input perturbations to be very small and use some notion of derivative.

The absolute condition number for "evaluate f at $x = x_0$" can be found by letting h be the 'change in input' and using Taylor's theorem: $f(x+h) = f(x) + hf'(x) + O(h^2)$. This says (for smooth f and small h) that

$$|f(x_0 + h) - f(x_0)| \approx |f'(x0)| \cdot |h|,$$

and

$$\frac{|f(x_0 + h) - f(x_0)|}{|f(x_0)|} \approx \left|\frac{x_0 f'(x_0)}{f(x_0)}\right| \cdot |\frac{h}{x_0}|,$$

giving $\nu = |f'(x_0)|$ and $\kappa = |\frac{x_0 f'(x_0)}{f(x_0)}|$.

The absolute condition number for "solve $f(x) = 0$" can be found by noticing that this problem is equivalent to "evaluate f^{-1} at $y = 0$".