Does this problem have a solution?

A problem is *well-posed* if it has a unique solution which depends continuosly on the data. Let's say that the data representing our problem is an element of some *input space* $\mathscr{D}$, and the solution is an element of some *output space* $\mathscr{R}$. The idea is very simple: our problem is well-posed if the mapping

$$f : \mathscr{D} \longrightarrow \mathscr{R}$$

is a continuous function. We certainly do not need an explicit representation of f (in fact, that is quite rare), but the idea of a well defined and continuous function helps us understand what we want from a problem and what it means to be a solution. The input space is the set of all instances of the problem we'd like to solve. For example, if our problem is to solve $Ax = b$, then a point in the input space is (A, b) consisting of a specific matrix A and a specific vector b.

The idea of well-posedness helps us clarify the relationship between our problem and our methods. For example, there remains ambiguity in the $Ax = b$ example above. If the input space is $\mathbb{R}^{m \times n} \times \mathbb{R}^m$, then we may have a unique solution, but there may be no solution, or there may be infinitely many; whereas if the input space is $\{A \in \mathbb{R}^{n \times n} : A^{-1} \text{ exists}\} \times \mathbb{R}^n$, then then every point in the input space represents a well-posed problem, and in this case we can explicitly write $f(A, b) = A^{-1}b$.

Suppose we want to compute the rank of an $n \times n$ matrix. Then $\mathscr{D} = \mathbb{R}^{n \times n}$, and since every square matrix has a well defined rank, our function $f(A) = rank(A)$ is well defined. Is f continuous? No. It is a discrete function of a continuous parameter, e.g. the matrix $E \in \mathbb{R}^{n \times n}$ with all entries zero except $e_{11} = \epsilon$, exhibits a discontinuity in f at 0, since $\lim_{\epsilon \to 0} |f(E) - f(0)| = 1 \neq f(0) = 0$. The fact that rank is ill-posed makes numerical linear algebra even more interesting!

On the other hand, $\det(A)$ is a continuous function on $\mathbb{R}^{n \times n}$. Hmmm... You will see that if some kind of singularity occurs for a specific parameter value α, then numerical analysis gets interesting for values *near* α. There are two problems exposed here. One is that $\det(A)$ is simply not a very good measure of rank deficiency. The other is that, like *rank*, *well-posedness* is a discrete function of continuous parameters: well-posedness is not well-posed!

So we've gone meta (and don't worry if you don't follow), but I have a very simple function in mind that is easy to evaluate (mostly): Let $\epsilon > 0$, and define

$$f_\epsilon(x) = \begin{cases} 0, & x \leq 0 \\ x/\epsilon, & 0 < x < \epsilon, \\ 1, & x \geq \epsilon \end{cases}$$

Certainly the problem "evaluate f_ϵ at x_0" is well-posed for any $x \in \mathbb{R}$, since f_ϵ is a continuous function on $\mathbb{R}$. But as the parameter ϵ gets smaller, f_ϵ gets closer to the *Heavyside* function, which is discontinuous at 0. Evaluation of the Heavyside function is illposed at $x = 0$. Therefore the question "is "evaluate $f_\epsilon(x)$ at $x = 0$" well-posed?" is illposed. In the face of modeling and rounding errors, etc., this binary idea of well-posed *vs.* illposed is too restrictive. We will replace it with the idea of *condition number*.