The Taylor Polynomial

We like polynomials mostly for their flexibility, simplicity, and smoothness. You know all about the simplicity: they are as smooth as you please, easy to differentiate and integrate, the polynomials of degree less than n form a vector space of dimension n, the product of polynomials is a polynomial, a polynomial of degree n has exactly n roots, etc., etc. An fundamental example of the flexibility of polynomials is the

Weierstrass Approximation Theorem:
If f is any function continuous over any finite inverval $[a, b]$, then for any $\epsilon > 0$ there is a polynomial p which satisfies $|p(x) - f(x)| < \epsilon$ for all $x \in [a, b]$.

If you don't yet appreciate this statement, draw a picture for this theorem (with an f that has corners if you're skeptical). We even have constructive proofs for this theorem (Bernstein polynomials). But this page is about the Taylor polynomial. The Taylor polynomial is not about approximating on an interval, but rather focusing locally, at a specific point. A statement is as follows:

Taylor Polynomial: If f has $n+1$ continuous derivatives on $[a, b]$, and $x_0 \in [a, b]$, then for each $x \in [a, b]$, there is a $\xi = \xi(x)$ between x and x_0 such that

$$f(x) = P_n(x) + R_n(x), \text{ where}$$

$$P_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j, \quad \text{and} \quad R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}.$$

$P_n = P_n(x; n, f, x_0)$ is the Taylor polynomial of degree n for f about x_0, and $R_n = R_n(x; n, f, x_0)$ is its remainder term (or *truncation error* term). P_n is the unique polynomial of degree n or less that satisfies

$$P_n^{(j)}(x_0) = f^{(j)}(x_0), \quad j = 0{:}n,$$

i.e., at x_0, P_n and its first n derivatives match f and its first n derivatives.

An equivalent way to write P_n is with the parameterization $x = x_0 + h$, giving

$$f(x_0 + h) \quad = \quad f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \cdots + \frac{h^n}{n!}f^{(n)}(x_0) \quad + \quad \frac{h^{n+1}}{(n+1)!}f^{(n+1)}(\xi).$$

The parameter ξ is an unknown parameter, and if that is troubling, we can write R_n as

$$R_n = \int_{x_0}^{x} \frac{f^{(n+1)}(s)}{n!}(s - x_0)^n \, ds.$$

However we write the error term, we can say that if f is smooth enough on $[a, b]$, there is a polynomial P_n, of degree $\leq n$, such that for all $x \in [a, b]$, (with $h = x - x_0$)

$$f(x) \;=\; P_n(x) \;+\; \mathrm{O}(h^{n+1}).$$