

## Quadrature

Numerical quadrature is the computation of the area (or volume, etc.) of a region bounded by one or more functions; in one variable it is the evaluation of

$$I \equiv \int_a^b f(x)dx.$$

Let's assume (i) that  $f$  is integrable on  $[a, b]$  and (ii) that we can evaluate  $f$  at any point in  $(a, b)$ . An important class of commonly used formulas for approximating  $I$  can be derived from the Lagrange interpolation result:

$$f(x) = P_n(x) + R_n(x), \quad \text{where } R(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^n (x - x_j),$$

$x_j \in (a, b)$ ,  $j = 0:n$ , and  $\xi = \xi(x) \in (a, b)$ .

Integrating gives

$$\int_a^b f(x)dx = \int_a^b P_n(x)dx + \int_a^b R_n(x)dx,$$

which yields a family of quadrature formulae

$$\int_a^b f(x)dx \approx \int_a^b P_n(x)dx = \sum_{i=0}^n c_i f(x_i),$$

where

$$c_i = \int_a^b L_{ni}(x)dx.$$

If  $f$  is smooth enough, the error in this approximation is  $\int_a^b R_n(x)dx$ .

If we discretize  $[a, b]$  into  $n+1$  equally spaced nodes, the integrals giving the  $c_i$  can be evaluated analytically *a priori*, to give quadrature rules. If we let  $h = (b - a)/n$  and  $x_j = a + jh$ ,  $j = 0:n$ , then a few of the *closed Newton-Cotes* rules are

$$\begin{aligned} \int_a^b f(x)dx &= \frac{b-a}{2}(f(a) + f(b)) - \frac{h^3}{12}f''(\xi_1) && \text{(Trapezoidal Rule, } n = 1) \\ &= \frac{h}{3}(f(x_0) + 4f(x_1) + f(x_2)) - \frac{h^5}{90}f^{(4)}(\xi_2) && \text{(Simpson's Rule, } n = 2) \end{aligned}$$

If we let  $h = (b - a)/(n+2)$ , and  $x_j = a + (j+1)h$ ,  $j = 0:n$ , we get the *open Newton-Cotes* formulae. This set of nodes does not include  $a$  or  $b$  and might be useful if there is a singularity at an endpoint:

$$\begin{aligned} \int_a^b f(x)dx &= \frac{b-a}{2}(f(x_0) + f(x_1)) + \frac{3h^3}{4}f''(\xi_1) && (n = 1) \\ &= \frac{b-a}{3}(2f(x_0) - f(x_1) + 2f(x_2)) + \frac{14h^5}{45}f^{(4)}(\xi_2) && (n = 2) \end{aligned}$$

The error terms have formulae that depend on the parity of  $n$  and whether open or closed, but they all look something like the closed rule with  $n$ -even:

$$R(f, n) = \frac{h^{n+3} f^{(n+2)}(\xi)}{(n+2)!} \int_0^n t \prod_{k=0}^n (t - k) dt, \quad \xi \in (a, b).$$