Quadratic interpolation (Introduction to Lagrange Interpolation)

Recall that if you know two points on a line, then you can find that line with the 2-point formula: given $(x_0, y_0), (x_1, y_1)$ on a line $p(x)$, then

$$p(x) = y_0 + \frac{x - x_0}{x_1 - x_0}(x - x_0) = y_0 \frac{x - x_1}{x_0 - x_1} + y_1 \frac{x - x_0}{x_1 - x_0}.$$

Let's try to do the same with 3 points on a parabola: find a quadratic polynomial that contains the points $(x_0, y_0), (x_1, y_1), (x_2, y_2)$. That is, find a_0, a_1, a_2 in

$$p(x) = a_0 + a_1 x + a_2 x^2 \text{ so that } p(x_i) = y_i, \; i = 0, 1, 2.$$

This gives the system of equations

$$\begin{aligned} a_0 + a_1 x_0 + a_2 x_0^2 &= y_0 \\ a_0 + a_1 x_1 + a_2 x_1^2 &= y_1 \\ a_0 + a_1 x_2 + a_2 x_2^2 &= y_2 \end{aligned}$$

with unknowns a_0, a_1, a_2 appearing linearly (a 3×3 linear system), which we can write as

$$Va = y, \text{ where}$$

$$V = \begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix}, \quad y = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}, \quad \text{and } a = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}.$$

This system has a unique solution if and only if V is nonsingular (has an inverse), and this is the case iff $\det(V) \neq 0$. It is not too hard to see that $\det(V) = (x_1 - x_0)(x_2 - x_1)(x_2 - x_0)$, and so (after solving $Va = y$ for the coefficients in the vector a) we will have a unique parabola passing through $(x_0, y_0), (x_1, y_1), (x_2, y_2)$, as long as $x_i \neq x_j$ for $i \neq j$.

We can find this same polynomial a different way: Define the 3 quadratic polynomials

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}, \quad L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}, \quad \text{and } L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}.$$

Notice that $L_0(x_0) = 1$ and $L_0(x_1) = L_0(x_2) = 0$, and in general

$$L_i(x_j) = 1, \text{ if } i = j \quad \text{and} \quad L_i(x_j) = 0, \text{ if } i \neq j.$$

Now define

$$p(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x)$$

and convince yourself that $p(x_i) = y_i, \; i = 0, 1, 2$. Thus, this parabola also contains the points $(x_0, y_0), (x_1, y_1), (x_2, y_2)$. Since the linear system $Va = y$ had a unique solution, this must be the same parabola as above, found without solving a system of equations.