Polynomials are Vectors

A very useful set of examples of finite dimensional vector spaces other than $\mathbb{F}^n$ are the spaces $\mathscr{P}_n(\mathbb{F})$ of polynomials of degree n or less with coefficients in $\mathbb{F}$ (for our purposes the field $\mathbb{F}$ will be $\mathbb{R}$ or $\mathbb{C}$). The standard ordered basis for $\mathscr{P}_n$ is $\mathscr{B} = \{1, x, \ldots, x^n\}$, and as such $\mathscr{P}_n$ is a vector space (over $\mathbb{F}$) of dimension $n+1$. You may also remember that $\mathscr{P}_n$ is isomorphic to $\mathbb{F}^{n+1}$. In fact, the natural isomorphism is the transformation that takes $p \in \mathscr{P}_n$ to its coordinate representation $[p]_{\mathscr{B}}$:

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n \in \mathscr{P}_n \iff [p]_{\mathscr{B}} = [a_0, a_1, a_2, \ldots, a_n]^T \in \mathbb{F}^{n+1}.$$

One of the reasons $\mathscr{P}_n$ is more interesting than $\mathbb{F}^{n+1}$ is that polynomials are well understood and well-behaved nonlinear functions. They are vectors which are also infinitely smooth functions. In fact, for any finite interval $[a, b] \subset \mathbb{R}$, any $p \in \mathscr{P}_n$ and any $r > 0$, the definite integral $\int_a^b |p(x)|^r \, dx$ exists. This allows us to bring some heavy mathematical machinery to bear on $\mathscr{P}_n$, and it allows us to define an inner product on $\mathscr{P}_n$, giving the inner product space $\mathscr{P}_n([a, b])$:

$$<p, q> \;=\; \int_a^b p(x)q(x) \, dx.$$

This inner product recognizes $\mathscr{P}_n$ as a space of functions. With an inner product comes *lengths* and *angles*. The *natural* norm on $\mathscr{P}_n([a, b])$ is

$$\|p\|_{[a,b]} \;=\; \sqrt{<p, p>} \;=\; \left(\int_a^b p(x)^2 \, dx \right)^{1/2}.$$

Two vectors, p and q in an inner product space are *orthogonal* if $<p, q>= 0$, so in this setting the polynomials p and q are orthogonal on $[a, b]$ if $\int_a^b p(x)q(x) \, dx = 0$. The basis vectors $\mathscr{B}$ above are not orthogonal on any interval, but we can apply the Gram-Schmidt process to find a basis $\mathscr{O} = \{P_0, P_1, \ldots, P_n\}$ which is orthogonal on, say $[-1, 1]$. These are the *Legendre polynomials*, they satisfy $<P_i, P_j>= 0$ if $i \neq j$, and $P(1) = 1$. Here is a recurrence relation:

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \quad \text{where } P_0(x) = 1 \text{ and } P_1(x) = x.$$

Why do we want an orthogonal basis? If $p = \sum_{i=0}^m a_i P_i$ with $<P_i, P_j>= 0$, then

$$\|p\|^2 = \int_a^b (\sum_{i=0}^n a_i P_i(x))^2 \, dx = \sum_{i=0}^n a_i^2 \|P_i\|^2.$$

Suppose you wanted to find the polynomial $p = \sum_{i=0}^m a_i P_i \in \mathscr{P}_n([-1, 1])$ closest to some function f, so that p is the minimizer of $g(a_0, a_1, \ldots, a_n) = \|f - \sum_{i=0}^m a_i P_i\|^2$. Calculus tells us that we find the a_i by setting the gradient of g to 0:

$$\frac{\partial g}{\partial a_i} \;=\; 0 \;=\; -2 <f, P_i> + 2a_i \|P_i\|^2, \text{ giving}$$

$$a_i = \frac{<f, P_i>}{\|P_i\|^2}.$$