Interpolating Polynomials

Suppose we are given a set of $n+1$ points in the xy-plane $(x_j, y_j),\ j = 0\!:\!n$ (called *knots*), with distinct *nodes* $x_j,\ j = 0\!:\!n$. Can we find a polynomial P which passes through (*interpolates*), these points? There are infinitely many, but only one of degree $\leq n$. Let's look at what we are asking. Write P as

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m.$$

Notice that we have $m+1$ coefficients available to satisfy our $n+1$ interpolation conditions, so hoping for uniqueness, let's try $m = n$ and see what happens.

The interpolation conditions are $P(x_j) = y_j,\ j = 0\!:\!n$, so for each j we get a linear equation in the a_j's:

$$a_0 + a_1 x_j + a_2 x_j^2 + \cdots + a_n x_j^n = y_j.$$

Writing these $n+1$ equations in $n+1$ unknowns in matrix form gives

$$Va = y,$$

where $a = (a_0, a_1, \ldots, a_n)^T$, $y = (y_0, y_1, \ldots, y_n)^T$, and $V = [x_i^j]_{i,j=0}^n$ is called a *Vandermonde* matrix. It is easy (but tedious) to show that the determininant of V is $\prod_{i>j}(x_i - x_j)$, so distinct nodes make V nonsingular. Therefore $Va = y$ has a unique solution representing the unique interpolating polynomial of degree $\leq n$.

There are many ways to represent this polynomial. We have just shown what it looks like in the standard ordered basis. There are other important polynomial bases for this problem. The Lagrange basis is the most natural. Let's define a set of polynomials

$$L_{nk}(x) = \prod_{i=0, i\neq k}^{n} \frac{x - x_i}{x_k - x_i}.$$

When restricted to the nodes, each of these polynomials is a Kronecker delta function:

$$L_{nk}(x_j) = \delta_{jk} = \begin{cases} 0 & , \quad j \neq k \\ 1 & , \quad j = k \end{cases}$$

We call the $L_{nk}(x)$ Lagrange basis functions, and the representation

$$P(x) = \sum_{i=0}^{n} y_i L_{ni}(x)$$

the *Lagrange form* of the interpolator. You check that $P(x_j) = y_j,\ j = 0\!:\!n$.

If the y's are function values ($y_i = f(x_i),\ 0\!:\!n$) for a function $f \in C^{(n+1)}([a, b])$ then we can write $f = P + R$, where R (below) is call the *truncation error*:

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n}(x - x_i), \qquad \xi \in (a, b).$$