Order of Convergence

The 'Big-O' notation is used to give an idea of the rate of convergence, but is often insufficient to convey how fast fast convergence can be. For quickly converging sequences, the *order of convergence* does a much better job. $\{p_n\} \to p$ of order α if there is a $\lambda > 0$ such that

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lambda.$$

The number λ is called the *asymptotic error constant.*

In the context of numerical methods, we usually think of $e_n \equiv p_n - p$ as an error ($\{e_n\} \to 0$), and we might write the definition above as

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^\alpha} = \lambda,$$

and for large enough n we should expect

$$|e_{n+1}| \approx \lambda |e_n|^\alpha.$$

It should be clear that if $p_n \to p$, then $e_n \to 0$, and thus $\alpha \geq 1$. (and if $\alpha = 1$, then $\lambda < 1$). The case $\alpha = 1$, $\lambda < 1$ corresponds to a exponential *rate* of convergence given by $\beta_n = \lambda^n = 1/(1/\lambda)^n$. This is a convergence rate that we thought was fast ($|p_n - p| = \mathrm{O}(\lambda^n)$, but we call it a *linear order* of convergence).

If $\alpha = 2$ and $\lambda = 1$, then for large n, $|e_{n+1}| \approx |e_n|^2$. For example, if $e_3 = 0.01$, then $e_4 \approx 0.0001$, $e_5 \approx 10^{-8}$, and $e_6 \approx 10^{-16}$. This is called a *quadratic* ($\alpha = 2$) order of convergence, and in this case the number of correct digits approximately doubles at each iteration. What about the number of correct digits in a cubically ($\alpha = 3$) convergent sequence?

If $\alpha > 1$, the order of convergence is called *superlinear* (in practice superlinear means $1 < \alpha < 2$). Superlinear convergence is exhibited by some very important methods, and we study it here a bit. The general definition of superlinear convergence of $\{p_n\} \to p$ is

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = 0.$$

This definition includes all cases where $\alpha > 1$ and also the case $\alpha = 1, \lambda = 0$.

Now superlinear convergence guarantees

$$\lim_{n\to\infty} \frac{|p_{n+1} - p_n|}{|p_n - p|} = \lim_{n\to\infty} \left| \frac{p_{n+1} - p + p - p_n}{p_n - p} \right| = \lim_{n\to\infty} \left| \frac{p_{n+1} - p}{p_n - p} + \frac{p - p_n}{p_n - p} \right| = 1.$$

Which says that for large enough n, we get a *computable* error estimate

$$|e_n| = |p_n - p| \approx |p_{n+1} - p_n|$$