Polynomials

A *polynomial* $p : \mathbb{F} \to \mathbb{F}$ is any function from a field $\mathbb{F}$ to $\mathbb{F}$ that can be written as

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

for all $x \in \mathbb{F}$, where n is a finite non-negative integer, and $a_i \in \mathbb{F}$. If the idea of a *field* is at all intimidating, don't worry: just replace $\mathbb{F}$ with the real numbers, $\mathbb{R}$, or the complex numbers, $\mathbb{C}$ (but there are lots of other fields). The highest power of x that appears is called the *degree* of p ($\deg(p)$). The polynomial above has degree n (if $a_n = 0$ we wouldn't include it in the expression), we call a polynomial *cubic* if $n = 3$, *quadratic* if $n = 2$, and *linear* if $n = 1$. We say a constant function $p(x) \equiv a_0$ has degree 0, and (if we have to) we say the zero function $p(x) \equiv 0$ has degree $-\infty$. If $a_n = 1$, we call p a *monic* polynomial.

Let's let $\mathscr{P} = \mathscr{P}(\mathbb{F})$ be the set of all polynomials (over $\mathbb{F}$), and $\mathscr{P}_n$ be the set of all polynomials with degree no more than n. Let $p, q \in \mathscr{P}_n$ and $a \in \mathbb{F}$. Then if we define $ap + q$ by $(ap + q)(x) = ap(x) + q(x)$, we find that $\mathscr{P}_n$ and $\mathscr{P}$ are vector spaces. $\mathscr{P}_n$ has dimension $n + 1$, and $\mathscr{P}$ is infinite dimensional. There are many types of *products* of polynomials that return polynomials, among them $(pq)(x) = p(x)q(x)$ and $(p \circ q)(x) = p(q(x))$. Ratios of polynomials are *not* generally polynomials (we call them rational functions), but if $d \neq 0$ is a polynomial with degree less than $\deg(p)$, there exist unique polynomials q and r, with $\deg(r) < \deg(d)$ such that

$$p(x) = d(x)q(x) + r(x).$$

Polynomials have some nice properties: The integral of a polynomial is a polynomial and the derivative of a polynomial is a polynomial; in fact if $C^k(S)$ is the set of all functions on S with k continuous derivatives, then $\mathscr{P} \subset C^\infty(\mathbb{C})$. *Smoooth...*

If p is monic with degree n, then there are $r_i \in \mathbb{C},\ i = 1, 2, \ldots, n$ such that

$$p(x) = (x - r_1)(x - r_2)\cdots(x - r_n).$$

If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then $\forall z \in \mathbb{F}$

$$p(z) = (\cdots((a_n z + a_{n-1})z + a_{n-2})z + \cdots a_1)z + a_0.$$

We can teach polynomials lots of tricks: If $x_0 \in \mathbb{F}$, and $y_i,\ i = 0{:}n$ is any sequence of $n + 1$ scalars, the *Taylor Polynomial* is the unique $P \in \mathscr{P}_n$ that satisfies

$$\frac{d^k}{dx^k}\{P(x)\}\big|_{x=x_0} = y_k, \quad k = 0{:}n.$$

If $(x_i, y_i),\ i = 0{:}n$ is such that $x_i \neq x_j$ for $i \neq j$, the *Lagrange interpolator* is the unique $P \in \mathscr{P}_n$ that satisfies

$$P(x_i) = y_i, \quad k = 0{:}n.$$

If f is *any* continuous real function on a finite interval $[a, b]$, then for any $\epsilon > 0$ there is a $P \in \mathscr{P}$ that satisfies

$$|f(x) - P(x)| < \epsilon, \quad \forall x \in [a, b].$$