Lagrange Interpolation on the Roots of Unity

Let n be a positive integer. Define $x_k = e^{2\pi ik/n}, \quad {\scriptstyle k=0:n-1}$, where $i = \sqrt{-1}$. Notice that $x_k^n = 1$, and as such these numbers are called the n^{th} *roots of unity*. They are evenly spaced around the *unit circle* $|z| = 1$ in $\mathbb{C}$: (De Moivre)

$$x_k = e^{2\pi ik/n} = (\cos(2\pi/n) + i\sin(2\pi/n))^k = \cos(2\pi k/n) + i\sin(2\pi k/n).$$

Now suppose we wish to interpolate the data $(x_k, y_k), \quad k = 0{:}n\text{–}1$ with a polynomial $p(x) = \sum_{j=0}^{n-1} a_j x^j$ of degree $n-1$ (the Lagrange interpolator).

The Vandermonde view says that p can be determined by solving the system

$$Va = y, \quad \text{where}$$

$$a = [a_0, a_1, \ldots, a_{n-1}]^T, \qquad y = [y_0, y_1, \ldots, y_{n-1}]^T \qquad \text{and}$$
$$e_k^t V = [1, x_k, x_k^2, \ldots, x_k^{n-1}].$$

Now let's investigate the (Hermitian) matrix $\overline{V}^T V = V^*V = [s_{kj}]$. If $k \neq j$,

$$s_{kj} = \sum_{r=0}^{n-1} \overline{x}_k^r x_j^r = \sum_{r=0}^{n-1} e^{2\pi(j-k)r/n} = \sum_{r=0}^{n-1} (e^{2\pi(j-k)/n})^r = \frac{(e^{2\pi(j-k)/n})^n - 1}{e^{2\pi(j-k)/n} - 1} = 0,$$

and for $k = j$,

$$s_{jj} = \sum_{r=0}^{n-1} 1^r = n.$$

so $V^*V = nI$. But then $V^*Va = V^*y$, so the coefficients of p are

$$a = V^{-1}y = (\frac{1}{n})V^*y.$$

What's the big deal? This is a *discrete Fourier transform* (DFT):

$$p(x) = \sum_{r=0}^{n-1} a_r x^r$$

is a frequency domain polynomial representation of the vector y, the a_r's are the DFT coefficients (we could write $\hat{y} = \mathcal{F}(y) = a$), and the interpolation conditions $p(x_k) = y_k$ simply say

$$y = \mathcal{F}^{-1}(\mathcal{F}(y)).$$

This DFT, as described here, is simply matrix multiplication by V^*, and requires $O(n^2)$ flops, but using divide-and-conquer to take advantage of the special structure of the roots of unity leads to

$$a = \text{FFT}(y),$$

requiring only $O(n\log(n))$ flops. Polynomials and trigonometric functions famously meet here, and at the Chebyshev polynomials, in some of the most fundamental and elegant of classical mathematics.