

Does this IVP have a solution?

$$y'(t) = f(t, y), \quad t \in [a, b], \quad y(a) = \alpha \quad (\text{IVP})$$

A problem is well posed if it has a unique solution which depends continuously on the data. In the case of (IVP) we can be more specific:

(IVP) is well posed if the perturbed problem

$$z'(t) = f(t, y) + \delta(t), \quad t \in [a, b], \quad z(a) = \alpha + \delta_0$$

has a unique solution which depends continuously on all $\delta \in C([a, b])$ and $\delta_0 \in \mathbb{R}$ at $\delta \equiv 0$ and $\delta_0 = 0$.

While we would rather have a condition number, we will accept well posedness as a surrogate for now. So do we have any sufficient conditions for (IVP) to be well posed? In fact, yes. Here is a simply stated theorem:

If $f(t, y)$ and $\frac{\partial f}{\partial y}(t, y)$ are continuous on $[a, b] \times \mathbb{R}$, then (IVP) is well posed.

There is an aesthetic weakness in this statement, but it captures the essence, and it is often a condition that we can check directly. We can often look at our mathematical model (or look into a subroutine to evaluate f) and check these conditions, and if f is smooth enough, (IVP) is well posed. If f does not satisfy these conditions, well posedness is unknown.

The weakness of that theorem is related to the weakness of well posedness relative to conditioning: There is a richer story to tell. In proving the theorem we need that $\forall (t, y_1), (t, y_2) \in [a, b] \times \text{Range}(y)$, $|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$, for some real L . This is guaranteed if f_y exists and is continuous in $[a, b] \times \mathbb{R}$, but the theorem misses those IVP's which are in between. Likewise, simply stating that the problem is well posed, misses those IVP's which are nearly illposed, i.e. those which are illconditioned.

A coarse first order absolute condition number for (IVP) might be

$$\nu = e^{(b-a)L}.$$

Yes, the difficulty can increase exponentially in time! If f changes rapidly wrt y over $[a, b]$, then we actually see this difficulty. Quite often the upper bound L is a gross overestimate of $f_y(t, y)$ for most of the solution, and this simple condition number is rather too pessimistic.