Example: High Order to First Order

An order m initial value problem is given in standard form as

$$y^{(m)}(t) = f(t, y, y', y'', \ldots, y^{(m-1)}), \;\; t \in [a, b], \;\; y^{(k)}(a) = \alpha_k, \;\; k = 0, 1, \ldots, m\text{–}1 \quad \text{(IVPm)}$$

By introducing the variables $u_1(t) = y(t), \;\; u_2(t) = y'(t), \;\; \ldots, \;\; u_m(t) = y^{(m-1)}(t)$, we have the first order system

$$\mathbf{u}'(t) = F(t, \mathbf{u}), \quad t \in [a, b], \quad \mathbf{u}(a) = \alpha, \qquad (IVP)$$

where

$$F(t, \mathbf{u}) \;\; = \;\; \begin{pmatrix} u_2(t) \\ u_3(t) \\ \vdots \\ u_m(t) \\ f(t, \mathbf{u}) \end{pmatrix}. \qquad (EF)$$

Here we give an example of this transformation. Suppose (IVPm) has the form

$$y'''(t) = 2\sin(y(t))y''(t) + (y'(t))^2 - 1/y(t) + t^2, \quad \text{with}$$

$$t \in [1, 3], \quad y(1) = 1, \quad y'(1) = -1, \quad y''(1) = 2.$$

Then $a = 1$ and $b = 3$. We define $u_1(t) = y(t), \;\; u_2(t) = y'(t)$, and $u_3(t) = y''(t)$. The vector form of $\mathbf{u}$ is

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix}, \quad \text{with} \quad \mathbf{u}(1) = \alpha = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

The (IVP) vector field is given by

$$\mathbf{u}'(t) = F(t, \mathbf{u}) = \begin{pmatrix} u_1'(t) \\ u_2'(t) \\ u_3'(t) \end{pmatrix} = \begin{pmatrix} u_2(t) \\ u_3(t) \\ 2\sin(u_1(t))u_3(t) + u_2^2(t) - 1/u_t(t) + t^2 \end{pmatrix}.$$

Make sure you understand how this is the same as (IVP) and (EF).

Euler's method (not a recommendation) for this system might look like

```
W(:,1) = α,  t = a,  h = (b-a)/N,
for j=1:N,
   W(:,j+1) = W(:,j) + h*F(t,W(:,j)) % feval('F',t,W(:,j)) or equiv.
   t = t + h
end
```