Adaptive Single-Step Methods

$$y'(t) = f(t,y), \quad t \in [a,b], \quad y(a) = \alpha \qquad \text{(IVP)}$$

How do we choose the step size, h, for an IVP solver? Typically we want h as large as possible but small enough to give sufficient accuracy. A simple and fairly safe (but not very efficient) approach is to guess an h and solve the problem two times: using step sizes of h and $h/2$, respectively. If the difference between the two simulations is small enough (on the coarser grid), then keep the finer simulation, otherwise go to $h/4$, etc.

If this sounds familiar, good. This is what the adaptive quadrature method did. But it did it automatically, and locally. The efficiency comes from the local adaptability, and the automatic step size decision is based on a *computable* error estimate. One construction of adaptive single-step methods is to compute an error estimate by taking a time step with an single-step method, say ϕ^*, with l.t.e. $\tau^* = \mathrm{O}(h^k)$, and a step with another single-step method, say ϕ, with l.t.e. $\tau = \mathrm{O}(h^{k+1})$, giving

$$w^*_{j+1} = w_j + h\phi^*(t_j, w_j) \qquad \text{and} \qquad w_{j+1} = w_j + h\phi(t_j, w_j).$$

The difference between these estimates guides a step size decision, as we discuss:

As usual, pretend that $w_j = y(t_j)$. Then $h\tau = y(t_{j+1}) - w_{j+1} = \mathrm{O}(h^{k+2})$, $h\tau^* = y(t_{j+1}) - w^*_{j+1} = \mathrm{O}(h^{k+1})$, and

$$\begin{aligned} h\tau^* &= y(t_{j+1}) - w^*_{j+1} \\ &= y_{j+1} - w_{j+1} + w_{j+1} - w^*_{j+1} \\ &= w_{j+1} - w^*_{j+1} + h\tau \\ &= w_{j+1} - w^*_{j+1} + \mathrm{O}(h^{k+2}) \end{aligned}.$$

This gives our computable error approximation $\tau^* \approx (w_{j+1} - w^*_{j+1})/h$.

Let's use this to compute a new step size qh. We will write $\tau = \tau(qh)$. Assuming h is small enough, there is a constant c^* such that $\tau^*(h) \approx c^* h^k$. Then $\tau^*(qh) \approx c^*(qh)^k = q^k c^* h^k \approx q^k \tau^*(h)$. If we have an error tolerance ϵ, then using our approximation above and solving $|\tau^*(qh)| \le \epsilon$ for q gives

$$q = \left(\frac{h\epsilon}{|w_{j+1} - w^*_{j+1}|} \right)^{1/k}.$$

If $q > 1$, then our estimate is saying that the step size could be bigger, so we would keep w_{j+1} and (possibly) increase the step size for the next step. If $q < 1$, then our estimate says that the step size should be smaller, so we throw out w_{j+1} and w^*_{j+1}, and do this step all over again with a new, smaller step size ($\le qh$).