Runge Kutta Methods of Order Two

$$y'(t) = f(t, y), \quad t \in [a, b], \quad y(a) = \alpha \qquad \text{(IVP)}$$

So the Runge-Kutta methods are single step methods that give us smaller errors than Euler, and more generality than the Taylor methods. We know that they sample f (the slope field) in the interval $[t, t + h]$ in order to approximate the average (*and ideal*) slope $(y(t + h) - y(t))/h$. *How* this is done is too much to cover in all of its (beautiful) generality, but we will explore the 2^{nd} order methods here.

Recall that the 2^{nd} order Taylor method is derived by dropping the $\mathrm{O}(h^3)$ term from

$$\begin{aligned} y(t + h) &= y(t) + hy'(t) + \tfrac{h^2}{2} y''(t) + \mathrm{O}(h^3) \\ &= y(t) + hf(t, y) + \tfrac{h^2}{2}[f_t(t, y) + f(t, y) f_y(t, y)] + \mathrm{O}(h^3). \end{aligned}$$

Thus, the Taylor iteration looks like

$$w_{j+1} = w_j + hf(t_j, w_j) + \frac{h^2}{2}[f_t(t_j, w_j) + f(t_j, w_j) f_y(t_j, w_j)]. \tag{1}$$

It is the f_t and f_y terms that restrict the general use of this method, so we will try to replace these. To that end we introduce the first order Taylor polynomial in two variables

$$f(t + \Delta_t, y + \Delta_y) = f(t, y) + \Delta_t f_t(t, y) + \Delta_y f_y(t, y) + \mathrm{O}(\Delta_t^2 + \Delta_t \Delta_y + \Delta_y^2). \tag{2}$$

To construct our method, we will sample the slope (f) at t_j (the Euler slope) and $t_j + \alpha h$, with $\alpha \in (0, 1]$. We then need to average these slopes in a way that will give $\mathrm{O}(h^2)$ l.t.e. Thus our method will look like

$$w_{j+1} = w_j + h[\lambda f(t_j, w_j) + (1 - \lambda) f(t_j + \alpha h, w_j + \alpha h f(t_j, w_j))].$$

Matching $f(t_j + \alpha h, w_j + \alpha h f(t_j, w_j))$ to $f(t + \Delta_t, y + \Delta_y)$ gives

$$\Delta_t = \alpha h \quad \text{and} \quad \Delta_y = \alpha h f(t_j, w_j).$$

Replacing $f(t_j + \Delta_t, w_j + \Delta_y)$ with the Taylor polynomial (2) gives (up to $\mathrm{O}(h^3)$)

$$\begin{aligned} w_{j+1} &= w_j + h[\lambda f(t_j, w_j) + (1 - \lambda)(f(t_j, w_j) + \alpha h(\ f_t(t_j, w_j) + f(t_j, w_j) f_y(t_j, w_j)\ )] \\ &= w_j + hf(t_j, w_j) + \alpha h^2 (1 - \lambda)[f_t(t_j, w_j) + f(t_j, w_j) f_y(t_j, w_j))]. \end{aligned}$$

Comparing this to the Taylor iteration (1), we see that $\alpha h^2(1 - \lambda) = h^2/2$, or

$$(1 - \lambda)\alpha = \frac{1}{2}.$$

Here, then, is the general form for all explicit second order Runge-Kutta methods:

$$w_{j+1} = w_j + \frac{h}{2\alpha}[\ (2\alpha - 1) f(t_j, w_j) + f(\ t_j + \alpha h,\ w_j + \alpha h f(t_j, w_j)\ )\ ].$$

Most authors include the formulas for $\alpha = \frac{1}{2}$ (the midpoint method), $\alpha = \frac{2}{3}$ (Heun's or Ralston's method), and $\alpha = 1$ (modified Euler (also called Heun's method)), but in fact there are a continuum of order 2 RK methods for $\alpha \in [\frac{1}{2}, 1]$.